HMMT 1998: Algebra Solutions

1. *Problem:* The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is \$23.50. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is \$39.50. What is the cost of 2 hamburgers, 2 milk shakes ,and 2 orders of fries at this restaurant?

Solution: Let H = hamburger, M = milk shake, and F = order of fries. Then $3H + 5M + F = \$23.50$. Multiplying the equation by 2 yields $6H + 10M + 2F = \$47$. Also, it is given that $5H + 9M + F = \$39.50$. Then subtracting the following equations

$$\begin{array}{llll} 6H & +10M & +2F & = \$47.00 \\ 5H & +9M & +F & = \$39.50 \end{array}$$

yields $H + M + F = \$7.50$. Multiplying the equation by 2 yields $2H + 2M + 2F = \boxed{\$15}$.

2. *Problem:* Bobbo starts swimming at 2 feet/s across a 100 foot wide river with a current of 5 feet/s. Bobbo doesn't know that there is a waterfall 175 feet from where he entered the river. He realizes his predicament midway across the river. What is the minimum speed that Bobbo must increase to make it to the other side of the river safely?

Solution: When Bobbo is midway across the river, he has travelled 50 feet. Going at a speed of 2 feet/s, this means that Bobbo has already been in the river for $\frac{50 \text{ feet}}{20 \text{ feet/s}} = 25$ s. Then he has traveled $5 \text{ feet/s} \cdot 25\text{s} = 125$ feet down the river. Then he has 175 feet-125 feet = 50 feet left to travel downstream before he hits the waterfall.

Bobbo travels at a rate of 5 feet/s downstream. Thus there are $\frac{50 \text{ feet}}{5 \text{ feet/s}} = 10\text{s}$ before he hits the waterfall. He still has to travel 50 feet horizontally across the river. Thus he must travel at a speed of $\frac{50 \text{ feet}}{10\text{s}} = 5$ feet/s. This is a $\boxed{3 \text{ feet/s}}$ difference from Bobbo's original speed of 2 feet/s.

3. *Problem:* Find the sum of every even positive integer less than 233 not divisible by 10.

Solution: We find the sum of all positive even integers less than 233 and then subtract all the positive integers less than 233 that are divisible by 10.

$2+4+\ldots+232 = 2(1+2+\ldots+116) = 116 \cdot 117 = 13572$. The sum of all positive integers less than 233 that are divisible by 10 is $10+20+\ldots+230 = 10(1+2+\ldots+23) = 2760$. Then our answer is 13572-2760 = $\boxed{10812}$.

4. *Problem:* Given that r and s are relatively prime positive integers such that $\frac{r}{s} = \frac{2(\sqrt{2}+\sqrt{10})}{5(\sqrt{3+\sqrt{5}})}$, find r and s.

Solution: Squaring both sides of the given equation yields $\frac{r^2}{s^2} = \frac{4(12+4\sqrt{5})}{25(3+\sqrt{5})} = \frac{16(3+\sqrt{5})}{25(3+\sqrt{5})} = \frac{16}{25}$. Because r and s are positive and relatively prime, then by inspection, $\boxed{r = 4 \text{ and } s = 5}$.

5. *Problem:* A man named Juan has three rectangular solids, each having volume 128. Two of the faces of one solid have areas 4 and 32. Two faces of another solid have areas 64 and 16. Finally, two faces of the last solid have areas 8 and 32. What is the minimum possible exposed surface area of the tallest tower Juan can construct by stacking his solids one on top of the other, face to face? (Assume that the base of the tower is not exposed).

Solution: Suppose that x, y, z are the sides of the following solids. Then Volume = $xyz = 128$. For the first solid, without loss of generality (with respect to assigning lengths to x, y, z), $xy = 4$ and $yz = 32$. Then $xy^2z = 128$. Then $y = 1$. Solving the remaining equations yields $x = 4$ and $z = 32$. Then the first solid has dimensions $4 \times 1 \times 32$.

For the second solid, without loss of generality, $xy = 64$ and $yz = 16$. Then $xy^2z = 1024$. Then $y = 8$. Solving the remaining equations yields $x = 8$ and $z = 2$. Then the second solid has dimensions $8 \times 8 \times 2$.

For the third solid, without loss of generality, $xy = 8$ and $yz = 32$. Then $y = 2$. Solving the remaining equations yields $x = 4$ and $z = 16$. Then the third solid has dimensions $4 \times 2 \times 16$.

To obtain the tallest structure, Juan must stack the boxes such that the longest side of each solid is oriented vertically. Then for the first solid, the base must be 1×4, so that the side of length 32 can

contribute to the height of the structure. Similarly, for the second solid, the base must be 8×2, so that the side of length 8 can contribute to the height. Finally, for the third solid, the base must be 4×2. Thus the structure is stacked, from bottom to top: second solid, third solid, first solid. This order is necessary, so that the base of each solid will fit entirely on the top of the solid directly beneath it.

All the side faces of the solids contribute to the surface area of the final solid. The side faces of the bottom solid have area $8 \cdot (8+2+8+2) = 160$. The side faces of the middle solid have area $16 \cdot (4+2+4+2) = 192$. The sides faces of the top solid have area $32 \cdot (4+1+4+1) = 320$.

Furthermore, the top faces of each of the solids are exposed. The top face of the bottom solid is partially obscured by the middle solid. Thus the total exposed area of the top face of the bottom solid is $8 \cdot 2 - 4 \cdot 2 = 8$. The top face of the middle solid is partially obscured by the top solid. Thus the total exposed area of the top face of the middle solid is $4 \cdot 2 - 4 \cdot 1 = 4$. The top face of the top solid is fully exposed. Thus the total exposed area of the top face of the top solid is $4 \cdot 1 = 4$.

Then the total surface area of the entire structure is $160 + 192 + 320 + 8 + 4 + 4 = \boxed{688}$.

6. *Problem:* How many pairs of positive integers (a, b) with $\leq b$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

Solution: $\frac{1}{a} + \frac{1}{b} = \frac{1}{6} \Rightarrow \frac{a+b}{ab} = \frac{1}{6} \Rightarrow ab = 6a+6b \Rightarrow ab-6a-6b = 0$. Factoring yields $(a-b)(b-6)-36 = 0$. Then $(a-6)(b-6) = 36$. Because a and b are positive integers, only the factor pairs of 36 are possible values of $a-6$ and $b-6$. The possible pairs are:

$$\begin{aligned} a-6 = 1,\ b-6 = 36 \\ a-6 = 2,\ b-6 = 18 \\ a-6 = 3,\ b-6 = 12 \\ a-6 = 4,\ b-6 = 9 \\ a-6 = 6,\ b-6 = 6 \end{aligned}$$

Because $a \leq b$, the symmetric cases, such as $a-6 = 12,\ b-6 = 3$ are not applicable. Then there are $\boxed{5}$ possible pairs.

7. *Problem:* Given that three roots of $f(x) = x^4+ax^2+bx+c$ are 2, -3, and 5, what is the value of $a+b+c$?

Solution: By definition, the coefficient of x^3 is negative the sum of the roots. In $f(x)$, the coefficient of x^3 is 0. Thus the sum of the roots of $f(x)$ is 0. Then the fourth root is -4. Then $f(x) = (x-2)(x+3)(x-5)(x+4)$. Notice that $f(1)$ is $1 + a + b + c$. Thus our answer is $f(1) - 1 = (1-2)(1+3)(1-5)(1+4) - 1 = \boxed{79}$.

8. *Problem:* Find the set of solutions for x in the inequality $\frac{x+1}{x+2} > \frac{3x+4}{2x+9}$ when $x \neq -2, x \neq \frac{9}{2}$.

Solution: There are 3 possible cases of x: 1) $-\frac{9}{2} < x$, 2) $\frac{9}{2} \leq x \leq -2$, 3) $-2 < x$. For the cases (1) and (3), $x+2$ and $2x+9$ are both positive or negative, so the following operation can be carried out without changing the inequality sign:

$$\begin{aligned} \frac{x+1}{x+2} &> \frac{3x+4}{2x+9} \\ \Rightarrow 2x^2+11x+9 &> 3x^2+10x+8 \\ \Rightarrow 0 &> x^2-x-1 \end{aligned}$$

The inequality holds for all $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$. The initial conditions were $-\frac{9}{2} < x$ or $-2 < x$. The intersection of these three conditions occurs when $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$.

Case (2) is $\frac{9}{2} \leq x \leq -2$. For all x satisfying these conditions, $x+2 < 0$ and $2x+9 > 0$. Then the following operations will change the direction of the inequality:

$$\frac{x+1}{x+2} > \frac{3x+4}{2x+9}$$
$$\Rightarrow 2x^2+11x+9 < 3x^2+10x+8$$
$$\Rightarrow 0 < x^2-x-1$$

The inequality holds for all $x < \frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2} < x$. The initial condition was $\frac{-9}{2} \leq x \leq -2$. Hence the intersection of these conditions yields all x such that $\frac{-9}{2} \leq x \leq -2$. Then all possible cases of x are $\boxed{\frac{-9}{2} \leq x \leq -2 \cup \frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}}$.

9. *Problem:* Suppose $f(x)$ is a rational function such that $3f(\frac{1}{x}) + \frac{2f(x)}{x} = x^2$ for $x \neq 0$. Find $f(-2)$.

Solution: Let $x = \frac{-1}{2}$. Then

$$3f(-2) + \frac{2f(\frac{-1}{2})}{\frac{-1}{2}} = \frac{1}{4}$$
$$\Rightarrow 3f(-2) - 4f(\frac{-1}{2}) = \frac{1}{4} \qquad (1)$$

Let $x = -2$. Then

$$3f(\frac{-1}{2}) + \frac{2f(-2)}{-2} = 4$$
$$\Rightarrow 3f(\frac{-1}{2}) - f(-2) = 4 \qquad (2)$$

Solving this system of equations $\{(1),(2)\}$ for $f(-2)$ yields $f(-2) = \boxed{\frac{67}{20}}$.

10. *Problem:* G.H. Hardy once went to visit Srinivasa Ramanujan in the hospital, and he started the conversation with: "I came here in taxi-cab number 1729. That number seems dull to me, which I hope isn't a bad omen." "Nonsense," said Ramanujan. "The number isn't dull at all. It's quite interesting. It's the smallest number that can be expressed as the sum of two cubes in two different ways." Ramanujan had immediately seen that $1729 = 12^3 + 1^3 = 10^3 + 9^3$. What is the smallest positive integer representable as the sum of the cubes of *three* positive integers in two different ways?

Solution: Let this smallest positive integer be represented as $a^3+b^3+c^3 = d^3+e^3+f^3$. By inspection, a solution is not possible with the first 4 cubes. We prove that it is impossible to write the same number as two different sums of the first 5 cubes. Because we necessarily need to use the 5th cube (otherwise, this proof would be for the first 4 cubes), we have $5^3+b^3+c^3 = d^3+e^3+f^3$. Without loss of generality, suppose $d = 5$. By inspection, there is no solution to $b^3+c^3 = e^3+f^3$, such that $b,c,e,f \leq 5$ and b,c and e,f are unique.

Then none of d,e,f are 5. Then at least two must be 4, otherwise the RHS would be too small. Without loss of generality, suppose $d = e = 4$. Then $b^3+c^3 = 3+f^3$. By inspection, there are no possible solutions if $b,c,f \leq 4$.

Thus if $a = 5$, there are no solutions.

Suppose that there is a solution within the first 6 cubes. Then $a = 6$. By the same analysis as above, $d = e = 5$, otherwise the RHS would be too small. Then $b^3+c^3 = 34+f^3$. By inspection, we see that a possible solution is $b = 3$, $c = 2$, $f = 1$. Then the desired integer is $6^3+3^3+2^3 = \boxed{251}$.