HMMT 1998: Calculus Solutions

1. *Problem:* Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after t minutes he is at position $(t, \sin t)$.

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for m minutes, his position is $(m, \cos t)$.

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

Solution: At arbitrary time t, Farmer Tim is at position $(t, \sin t)$ and Alex is at position $(t-5, \cos t)$. Hence at time t, the distance, d, between Tim and Alex is $d = \sqrt{(\sin t - \cos t)^2 + 25}$. To find the maximum value of d, we solve for t such that $\frac{dd}{dt} = 0$.

$\frac{dd}{dt} = \frac{(\sin t - \cos t)(\cos t + \sin t)}{\sqrt{(\sin t - \cos t)^2 + 25}}$. Then $\frac{dd}{dt} = 0 \Rightarrow \sin^2 t - \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t$. Equality happens if t is any constant multiple of $\frac{\pi}{4}$.

Notice that to maximize d, we need to maximize $(\sin t - \cos t)^2$. This is achieved when $\cos t = -\sin t$. Because we determined earlier that t is a constant multiple of $\frac{\pi}{4}$, then with this new condition, we see that t must be a constant multiple of $\frac{3\pi}{4}$.

Then $(\sin t - \cos t)^2 = 2 \Rightarrow d = \boxed{\sqrt{29}}$.

2. *Problem:* A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1m. If the water level in the cube is falling at a rate of 1 cm/s, at what rate is the water level in the cylinder rising?

Solution: The magnitude of the change in volume per unit time of the two solids is the same. The change in volume per unit time of the cube is 1 cm$\cdot m^2$/s. The change in volume per unit time of the cylinder is $\pi \cdot \frac{dh}{dt} \cdot m^2$, where $\frac{dh}{dt}$ is the rate at which the water level in the cylinder is rising.

Solving the equation $\pi \cdot \frac{dh}{dt} \cdot m^2 = 1 \text{ cm} \cdot m^2/\text{s}$ yields $\boxed{\frac{1}{\pi} \text{ cm/s}}$.

3. *Problem:* Find the area of the region bounded by the graphs of $y = x^2$, $y = x$, and $x = 2$.

Solution: There are two regions to consider. First, there is the region bounded by $y = x^2$ and $y = x$, in the interval $[0, 1]$. In this interval, the values of $y = x$ are greater than the values of $y = x^2$, thus the area is calculated by $\int_0^1 (x - x^2)\, dx$.

Second, there is the region bounded by $y = x^2$ and $y = x$ and $x = 2$, in the interval $[1, 2]$. In this interval, the values of $y = x^2$ are greater than the values of $y = x$, thus the area is calculated by $\int_1^2 (x^2 - x)\, dx$.

Then the total area of the region bounded by the three graphs is $\int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx = \boxed{1}$.

4. *Problem:* Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$, for $-1 \leq x \leq 1$. Find $\sqrt{e^{\int_0^1 f(x) dx}}$.

Solution: Observe that $f(x)$ is merely an infinite geometric series. Thus $f(x) = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$. Then $\int_0^1 \frac{2}{2-x} = 2 \ln 2$. Then $\sqrt{e^{2 \ln 2}} = \sqrt{2^2} = \boxed{2}$.

5. *Problem:* Evaluate $\lim_{x \to 1} x^{\frac{x}{\sin(1-x)}}$.

Solution: Rewrite the expression to evaluate as $e^{\ln x^{\frac{x}{\sin(1-x)}}}$. Then we must evaluate $\lim_{x \to 1} e^{\ln x^{\frac{x}{\sin(1-x)}}}$.

$\lim_{x \to 1} \ln x^{\frac{x}{\sin(1-x)}} = \lim_{x \to 1} \left(\frac{x}{\sin(1-x)} \ln x \right)$. Because direct calculation of the limit results in indeterminate form $(\frac{1}{0} \cdot 0)$, we can use L'Hopital's rule to evaluate the limit. By L'Hopital's rule, $\lim_{x \to 1} \left(\frac{x}{\sin(1-x)} \ln x \right) = \lim_{x \to 1} \frac{\ln x + 1}{-\cos(1-x)}$. This limit is simply -1.

Hence $\lim_{x\to 1} e^{\ln x^{\frac{x}{\sin(1-x)}}} = e^{-1} = \boxed{\frac{1}{e}}$.

6. *Problem:* Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place 1,875 feet from the prison and was spotted by a guard with a crossbow. The guard fired an arrow with an initial velocity of 100 ft/s. At the same time, Edward started running away with an acceleration of 1 ft/s^2. Assuming that air resistance causes the arrow to decelerate at 1 ft/s^2 and that it does hit Edward, how fast was the arrow moving at the moment of impact (in ft/s)?

Solution: We use the formula for distance, $d = \frac{1}{2}at^2 + vt + d_0$. Then after t seconds, Edward is at location $1875 + \frac{1}{2}(1)(t^2)$ from the prison. After t seconds, the arrow is at location $\frac{1}{2}(-1)(t^2) + 100t$ from the prison. When the arrow hits Edward, both objects are at the same distance away from the tower. Hence $1875 + \frac{1}{2}(1)(t^2) = \frac{1}{2}(-1)(t^2) + 100t$. Solving for t yields $t^2 - 100t + 1875 = 0 \Rightarrow t = 25$ or $t = 75$. Then it must be $t = 25$, because after the arrow hits Edward, he will stop running.

After 25 seconds, the arrow is moving at a velocity of 100-25(1) $= \boxed{75 \text{ ft/s}}$.

7. *Problem:* A parabola is inscribed in equilateral triangle ABC of side length 1 in the sense that AC and BC are tangent to the parabola at A and B, respectively. Find the area between AB and the parabola.

Solution: Suppose $A = (0,0)$, $B = (1,0)$, and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then the parabola in question goes through $(0,0)$ and $(1,0)$ and has tangents with slopes of $\sqrt{3}$ and $-\sqrt{3}$, respectively, at these points. Suppose the parabola has equation $y = ax^2 + bx + c$. Then $\frac{dy}{dx} = 2ax + b$.

At point $(0,0)$, $\frac{dy}{dx} = b$. Also the slope at $(0,0)$, as we determined earlier, is $\sqrt{3}$. Hence $b = \sqrt{3}$. Similarly, at point $(1,0)$, $\frac{dy}{dx} = 2a + b$. The slope at $(1,0)$, as we determined earlier, is $-\sqrt{3}$. Then $a = -\sqrt{3}$.

Since the parabola goes through $(0,0)$, $c = 0$. Hence the equation of the parabola is $y = -\sqrt{3}x^2 + \sqrt{3}x$. The desired area is simply the area under the parabolic curve in the interval $[0,1]$.

Hence $\int_0^1 \left(-\sqrt{3}x^2 + \sqrt{3}x\right) dx = \boxed{\frac{\sqrt{3}}{6}}$.

8. *Problem:* Find the slopes of all lines passing through the origin and tangent to the curve $y^2 = x^3 + 39x - 35$.

Solution: Any line passing throug the origin has equation $y = mx$, where m is the slope of the line. If a line is tangent to the given curve, then at the point of tangency, (x,y), $\frac{dy}{dx} = m$.

First, we calculate $\frac{dy}{dx}$ of the curve: $2ydy = 3x^2dx + 39dx \Rightarrow \frac{dy}{dx} = \frac{3x^2+39}{2y}$. Substituting mx for y, we get the following system of equations:

$$m^2x^2 = x^3 + 39x - 35$$
$$m = \frac{3x^2 + 39}{2mx}$$

Solving for x yields the equation $x^3 - 39x + 70 = 0 \Rightarrow (x-2)(x+7)(x-5) = 0 \Rightarrow x = 2$ or $x = -7$ or $x = 5$. These solutions indicate the x-coordinate of the points at which the desired lines are tangent to the curve. Solving for the slopes of these lines, we get $m = \pm\frac{\sqrt{51}}{2}$ for $x = 2$, no real solutions for $x = -7$, and $m = \pm\frac{\sqrt{285}}{5}$ for $x = 5$. Thus $\boxed{m = \pm\frac{\sqrt{51}}{2}, \pm\frac{\sqrt{285}}{5}}$.

9. *Problem:* Evaluate $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$.

Solution: Note that if we take the integral of $f(x)$ in problem 4, we get the function $F(x) = x + \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 2^2} + \ldots$. Evaluating this integral in the interval $[0,1]$, we get $1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} + \ldots$, which is the desired sum.

Hence $\int_0^1 \frac{2}{2-x} dx = 2\ln 2$.

10. *Problem:* Let S be the locus of all points (x, y) in the first quadrant such that $\frac{x}{t} + \frac{y}{1-t} = 1$ for some t with $0 < t < 1$. Find the area of S.

Solution: Solving for t in the given equation, we get $t^2 + (y - x - 1)t + x = 0$. Using the quadratic equation, we get $t = \frac{(x+1-y)\pm\sqrt{(y-x-1)^2-4x}}{2}$. For all valid combinations of (x, y), t is positive and less than 1 (this is easy to see by inspection). All valid combinations of (x, y) are those that make $(y - x - 1)^2 - 4x \geq 0$.

Solving for y in the equation $(y-x-1)^2-4x = 0$ yields $y^2-(2x+2)y+(x-1)^2 \geq 0 \Rightarrow y = (x+1)\pm 2\sqrt{x}$. In the original equation, it is given that $\frac{x}{t} + \frac{y}{1-t} = 1$, and $0 < t < 1$. This implies that $x, y < 1$. Then the only possible $y < 1$ that satisfies $(y - x - 1)^2 - 4x = 0$ is $y = x + 1 - 2\sqrt{x}$.

Then to satisfy the inequality $(y - x - 1)^2 - 4x \geq 0$, we must have $y \leq x + 1 - 2\sqrt{x}$. Recall that this is when $0 < y < 1$. Hence we integrate in the interval $[0, 1]$: $\in_0^1 x + 1 - 2\sqrt{x} = \boxed{\frac{1}{6}}$.