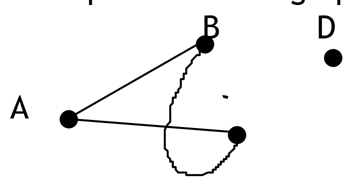
Power Question - Coloring Graphs

Perhaps you have heard of the Four Color Theorem (if not, don't panic!), which essentially says that any map (e.g. a map of the United States) can be colored with four or fewer colors without giving neighboring regions (e.g. Massachusetts and New Hampshire) the same color. It was, until 22 years ago, one of the most famous unsolved problems in mathematics. The only known proof, however, is so long that it requires a computer to carry it out. This power question will lead you through the basic definitions and theorems of graph theory necessary to prove the Five Color Theorem, a weaker and much easier (though still quite challenging) statement.

Part I - Graphs

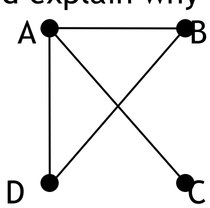
Definitions: A *graph* is a collection of points and lines (or curves), called *vertices* and *edges*, respectively, where each edge connects exactly two distinct vertices and any two vertices are connected by at most one edge. We say that two vertices are *adjacent* if they are connected by an edge.
Notation: For this problem we will denote vertices by capital letters and the edge connecting vertices X and Y by XY. Vertices will be drawn as dots so that they will be distinguishable from edge crossings. We will denote graphs by capital script letters, usually G or H.

Example: Here is a graph with vertices A, B, C, D, and edges AB, AC, BC.



a. Which of the following are graphs? List the vertices and edges of each graph and explain why the others are not graphs. [1 point each]

i.



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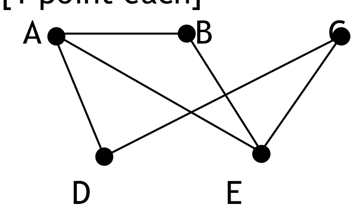

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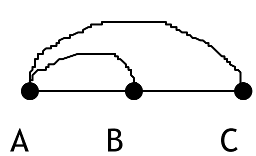

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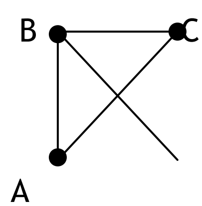

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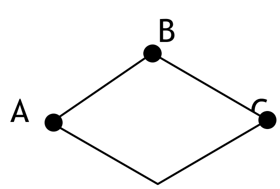

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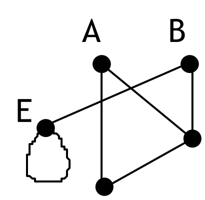
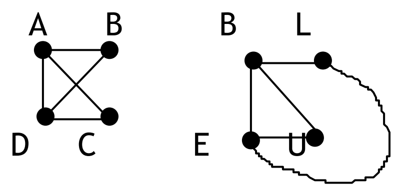


C

D

Definition: Graph G is said to be *isomorphic* (from Greek roots meaning “same structure”) to H, written G ≅ H, if the vertices of G can be given the names of the vertices of H in such a way that G and H have the same edges. This relabling is called an *isomorphism* and is reversible, thus implying H ≅ G as well.

Example:



G H

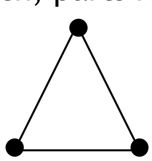
G ≅ H since we can relable the vertices of G as follows:
A↔B, B↔L, C↔U, D↔E
(we write the double arrows to emphasize the reversibility of the isomorphism).

b. [1 point each]
i. List the edges of H.
ii. Show that H ≅ G in the example by writing an isomorphism from H to G.

From now on the diagrams in the problem will not have the vertices explicitly labled unless necessary. Thus to show two unlabled graphs are isomorphic we can arbitrarily lable the vertices of one and then show that the other can be labled with the same names to yield the same edges.

c. For each of the following pairs of graphs, state whether or not they are isomorphic (you do not need to justify your answer). [parts *i*-*iii* are 1 point each, parts *iv*-*vi* are 3 points each]

i.



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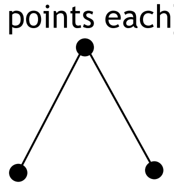

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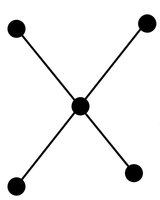

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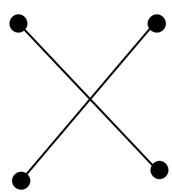

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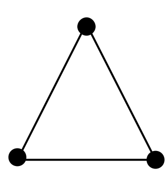

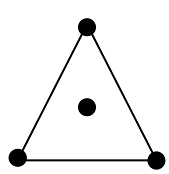

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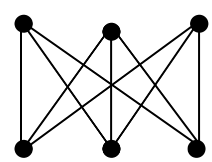

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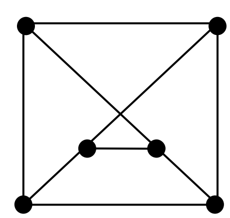

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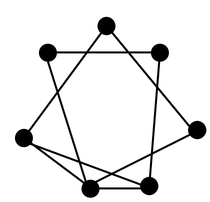

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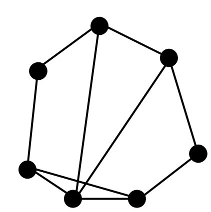

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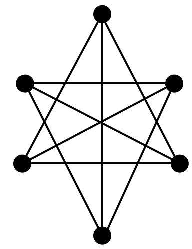

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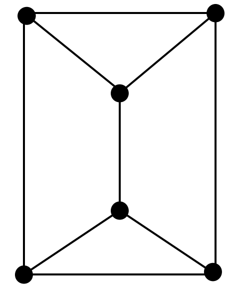
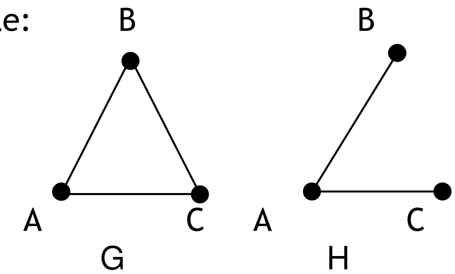


A couple more definitions are best given in this section, though there won't be any questions specifically about them until later.

Definitions: A graph H is a *subgraph* of a graph G if every vertex of H is a vertex of G and every edge of H is an edge of G.

The *degree* of a vertex is the number of edges connected to it.

Example:



H is a subgraph of G. Vertex B has degree 2 in G and degree 1 in H.

<u>Part II - Planar Graphs</u>

There are not many nontrivial properties possessed by all graphs, so we will restrict our attention to graphs with certain nice properties that we can exploit to prove cool theorems. So now we need one more round of definitions, then the real fun begins!

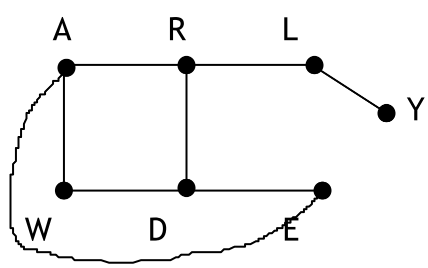
Definitions: A graph is *planar* if it is isomorphic to a graph that has been drawn in a plane without edge crossings. Otherwise a graph is nonplanar. It shouldn't be too hard to see that every subgraph of a planar graph is planar.

A *walk* in a graph is a sequence A_1 A_2 ... A_n of not necessarily distinct vertices in which A_k is adjacent to A_{k+1} for k = 1, 2, ..., n-1.

A graph is *connected* if every pair of vertices is joined by a walk, or equivalently if there is a walk that passes through every vertex at least once. Otherwise a graph is said to be disconnected.

When a planar graph is actually drawn in a plane without edge crossings, it cuts the plane into regions called *faces* of the graph.

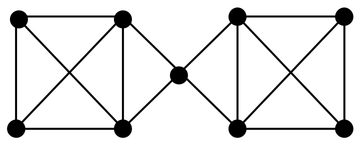
Example:



Is connected since the walk EDWARDEARLY passes through every vertex at least once. It has 3 faces (don't forget the exterior region when counting).

a. How many vertices, edges, and faces do each of the following graphs have? [3 points each]

i.



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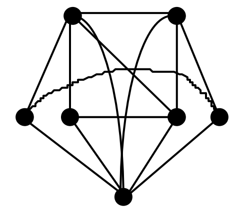
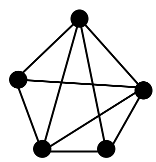


b. For each of the following graphs, state whether or not it is planar. You do not need to justify your answer. [3 points each]

i.



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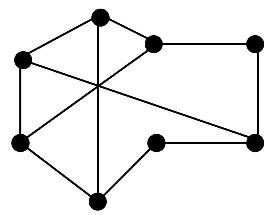

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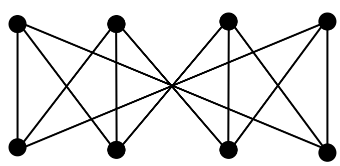

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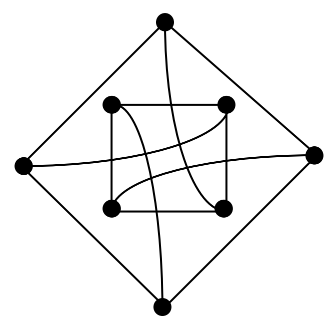
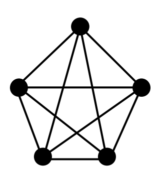


Euler's formula states if a connected planar graph has v vertices, e edges, and f faces, then v-e+f=2. Proving this would go beyond the scope of this problem, so you may take this formula as given.

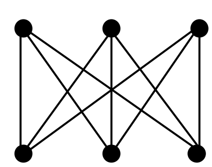
c. [15 points] Prove that if G is planar and connected with v≥3, then $\frac{3}{2}f \le e \le 3v - 6$.

d. Prove that the following graphs are nonplanar.

i. [5 points]



ii. [10 points]

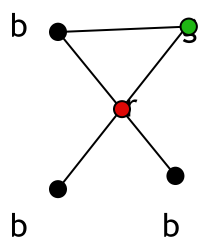


e. [12 points] Prove that every planar graph has at least one vertex of degree 5 or less.

<u>Part III - Coloring</u>

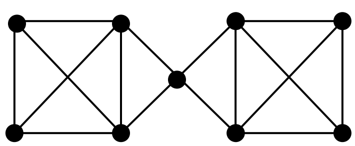
Now we can begin talking about coloring planar graphs.
Definition: We say a graph has been *colored* if a color has been assigned to each vertex in such a way that adjacent vertices have different colors.
The *chromatic number* χ (the Greek letter chi, pronounced like sky without the s) of a graph is the smallest number of colors with which it can be colored.

Example: Here is a graph with chromatic number 3, colored in black, green, and red.

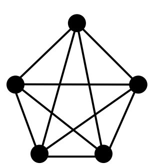


a. Calculate χ for each of the following graphs. [3 points each]

i.



ii.



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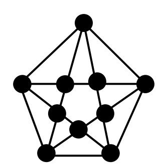

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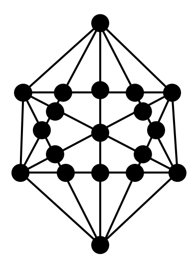
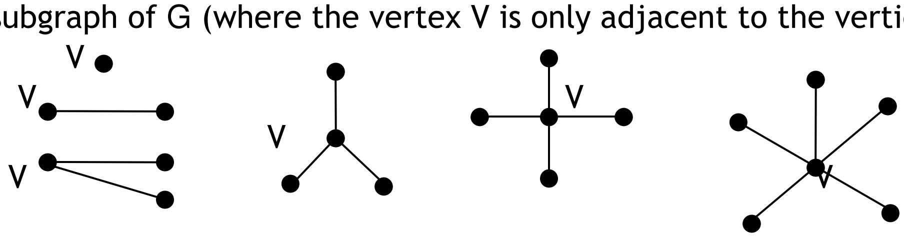


b. [14 points] Given a graph G, let $\bar{\chi}$ be the chromatic number of the graph obtained by taking the vertices of G and drawing edges only between those that are not adjacent in G. Prove that $\chi + \bar{\chi} \geq 2\sqrt{v}$.

c. Assume G is a planar graph with χ>5.
i. [5 points] Prove that at least one of the following must be isomorphic to a subgraph of G (where the vertex V is only adjacent to the vertices shown).



ii. [20 points] Prove that if one of the above is a subgraph of G (where the vertex V is only adjacent to the vertices shown) then we can remove one vertex (and all edges touching it) from G to obtain a graph with fewer vertices which also has $\chi>5$ (you may use the Jordan Curve Theorem, which states that a closed loop that does not intersect itself divides the plane into an inside and outside, and if a continuous curve joins a point on the inside to one on the outside then it must cross the loop).
iii. [5 points] Prove the Five Color Theorem: Every planar graph has $\chi\leq5$.