Advanced Topics
Rice Mathematics Tournament 2000

1. How many different ways are there to paint the sides of a tetrahedron with exactly 4 colors? Each side gets its own color, and two colorings are the same if one can be rotated to get the other.

2. Simplify $\left(\frac{-1+i\sqrt{3}}{2}\right)^6 + \left(\frac{-1-i\sqrt{3}}{2}\right)^6$ to the form $a + bi$.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

4. Five positive integers from 1 to 15 are chosen without replacement. What is the probability that their sum is divisible by 3?

5. Find all 3-digit numbers which are the sums of the cubes of their digits.

6. 6 people each have a hat. If they shuffle their hats and redistribute them, what is the probability that exactly one person gets their own hat back?

7. Assume that a, b, c, d are positive integers, and $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2} = 15$. Find $ac + bd - ad - bc$.

8. How many non-isomorphic graphs with 9 vertices, with each vertex connected to exactly 6 other vertices, are there? (Two graphs are isomorphic if one can relabel the vertices of one graph to make all edges be exactly the same.)

9. The Cincinnati Reals are playing the Houston Alphas in the last game of the Swirled Series. The Alphas are leading by 1 run in the bottom of the 9th (last) inning, and the Reals are at bat. Each batter has a $\frac{1}{3}$ chance of hitting a single and a $\frac{2}{3}$ chance of making an out. If the Reals hit 5 or more singles before they make 3 outs, they will win. If the Reals hit exactly 4 singles before making 3 outs, they will tie the game and send it into extra innings, and they will have a $\frac{3}{5}$ chance of eventually winning the game (since they have the added momentum of coming from behind). If the Reals hit fewer than 4 singles, they will LOSE! What is the probability that the Alphas hold off the Reals and win, sending the packed Alphadome into a frenzy? Express the answer as a fraction.

10. I call two people A and B and think of a natural number n. Then I give the number n to A and the number $n + 1$ to B. I tell them that they have both been given natural numbers, and further that they are consecutive natural numbers. However, I don't tell A what B's number is and vice versa. I start by asking A if he knows B's number. He says "no". Then I ask B if he knows A's number, and he says "no" too. I go back to A and ask, and so on. A and B can both hear each other's responses. Do I ever get a "yes" in response? If so, who responds first with "yes" and how many times does he say "no" before this? Assume that both A and B are very intelligent and logical. You may need to consider multiple cases.