Advanced Topics Solutions
Rice Mathematics Tournament 2000

1. Assume we have 4 colors - 1, 2, 3, and 4. Fix the bottom as color 1. On the remaining sides you can have colors 2, 3, 4 (in that order), or 2, 4, 3, which are not rotationally identical. So, there are **2** ways to color it.

2. Since $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, we can write the first term as $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^6$. Since $\cos\frac{4\pi}{3} = -\frac{1}{2}$ and $\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, we can write the second term as $\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^6$. Now, we apply DeMoivre's Theorem to simplify the first expression to $(\cos 6\cdot\frac{2\pi}{3} + \sin 6\cdot\frac{2\pi}{3}) = (\cos 4\pi + \sin 4\pi) = 1+0 = 1$. Similarly, we simplify the second expression to $(\cos 6\cdot\frac{4\pi}{3} + \sin 6\cdot\frac{4\pi}{3}) = (\cos 8\pi + \sin 8\pi) = 1 + 0 = 1$. Thus, the total sum is $1 + 1 =$ **2**.

3. We know that $\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{n}-\frac{1}{n+2}}{2}$. So, if we sum this from 1 to ∞, all terms except for $\frac{\frac{1}{1}}{2} + \frac{\frac{1}{2}}{2}$ will cancel out (a "telescoping" series). Therefore, the sum will be $\mathbf{\frac{3}{4}}$.

4. The possibilities for the numbers are:

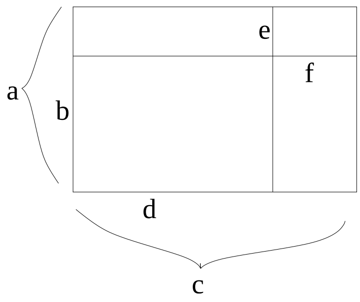
   - all five are divisible by 3
   - three are divisible by 3, one is $\equiv 1 \pmod 3$ and one is $\equiv 2 \pmod 3$
   - two are divisible by 3, and the other three are either $\equiv 1 \pmod 3$ or $\equiv 2 \pmod 3$
   - one is divisible by 3, two are $\equiv 1 \pmod 3$ and two are $\equiv 2 \pmod 3$
   - four are $\equiv 1 \pmod 3$ and one is $\equiv 2 \pmod 3$
   - four are $\equiv 2 \pmod 3$ and one is $\equiv 1 \pmod 3$

   This gives us 1001 possible combinations out of $\binom{15}{5}$ or 3003. So, the probability is $\frac{1001}{3003} = \mathbf{\frac{1}{3}}$.

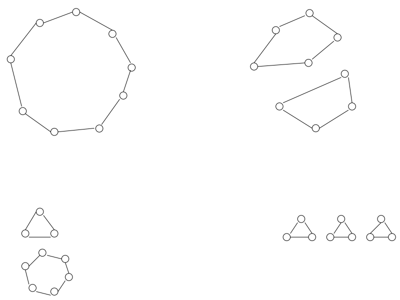
5. **153,370,371,407**

6. There are 6 people that could get their hat back, so we must multiply 6 by the number of ways that the other 5 people can arrange their hats such that no one gets his/her hat back. So, the number of ways this will happen is $(6 \cdot$ derangement of $5)$, or $6 * 44 = 264$. Since there are $6! = 720$ possible arrangements of hats, the probability of exactly one person getting their hat back is $\frac{264}{720} = \mathbf{\frac{11}{30}}$.

7. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2+f^2} = 15$ (since this is $\sqrt{a^2+c^2} - \sqrt{b^2+d^2}$). Also, note that $ac + bd - ad - bc = (a-b)(c-d) = ef$. So, solving for e and f, we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2+4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4}\cdot 12 = 9$. Therefore, $\boldsymbol{ef} = 9 * 12 =$ **108**.



8. It suffices to consider the complements of the graphs, so we are looking for graphs with 9 vertices, where each vertex is connected to 2 others. There are **4** different graphs - see below.



9. The probability of the Reals hitting 0 singles is $(\frac{2}{3})^3$. The probability of the Reals hitting exactly 1 single is $\binom{3}{2}\cdot(\frac{2}{3})^3\cdot\frac{1}{3}$, since there are 3 spots to put the two outs (the last spot must be an out, since the inning has to end on an out). The probability of the Reals hitting exactly 2 singles is $\binom{4}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^3$. The probability of the Reals hitting exactly 3 singles is $\binom{5}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^3$. If any of these happen, the Alphas win right away. Adding these gives us a $\frac{656}{729}$ chance of this happening. If exactly 4 singles occur (with probability $\binom{6}{2}\cdot(\frac{2}{3})^3\cdot(\frac{1}{3})^4$), then there is a $\frac{2}{5}$ chance that the Alphas win. The probability of this happening is $\frac{2}{5}\cdot\frac{40}{729}$. Thus, the total probability of the Alphas winning is the sum of these two probabilities, or $\frac{656}{729}+\frac{16}{729}=\mathbf{\frac{224}{243}}$.
10. A will say yes when B says no to $n-1$ or n, as A will then know B's number is one greater than A's number. Thus, A responds first, after $\boldsymbol{\frac{n-1}{2}}$ **"no" responses if n is odd**, after $\boldsymbol{\frac{n}{2}}$ **"no" responses if n is even**.