Algebra Solutions
Rice Mathematics Tournament 2000

1. The only integers that satisfy $|x| + 5 < 7$ are the ones that satisfy $|x| < 2$ - namely, $-1, 0, 1$. The integers that satisfy $|x - 3| > 2$ are $6, 7, 8, \ldots$ and $0, -1, -2, \ldots$. So, the integers that satisfy both are $0, -1$, and there are **2** of them.

2. $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$ can be factored into $(2000 - 1999)2000^2 + 1999^2(-2000 + 1999)$, which reduces to $2000^2 - 1999^2$. This factors into $(2000 + 1999)(2000 - 1999)$ which is equal to **3999**.

3. Let the scores be a,b,c,d,e, where $0 \leq a \leq b \leq c \leq d \leq e \leq 100$. So, the mean is $\frac{1}{5}(a + b + c + d + e)$, and the median is c. So, we want to maximize $\frac{1}{5}(a + b + c + d + e) - c$. To do this, we must maximize d and e and minimize or maximize c. One way to do this is to let $a = b = c = 0$ and $d = e = 100$, so the difference between the mean and the median is $\frac{1}{5}(0 + 0 + 0 + 100 + 100) - 0) = \frac{200}{5} = \mathbf{40}$. If we maximize c, then $c = d = e = 100$, and then the mean is $\frac{1}{5}(0 + 0 + 100 + 100 + 100) = 60$, and the median is 60, with a difference of 40 as well.

4. The price starts at \$100. Clearly, the order of price changes does not matter. It is reduced by 10% three times (\$100 $\rightarrow$ \$90 $\rightarrow$ \$81 $\rightarrow$ \$72.90), and the new price is \$72.90. It is increased by 10% four times (\$72.90 $\rightarrow$ \$80.19 $\rightarrow$ \$88.209 $\rightarrow$ \$97.0299 $\rightarrow$ \$106.73289) , and the new price is \$106.73289. Rounded to the nearest cent, this is **\$106.73**.

5. Every day Edward works, he gets $\frac{1}{9}$ of the test done. Similarly, every day Barbara works, she gets $\frac{1}{10}$ of the test done, every day Abhinav works, he gets $\frac{1}{11}$ of the test done, and every day Alex works, he gets $\frac{1}{12}$ of the test done. So, after 4 days (after everyone has worked on the test one day, they have completed $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} = 38.535\%$ of the test. After 8 days, they have completed twice that, or 77.0707% of the test. After Edward, Barbara, and Abhinav each work one more day, the test will be complete in the minimum amount of time, so the test will take **11 days** to complete. If the least efficient workers work after the 8th day, the test still takes 11 days to complete.

6. A shortest path is $x \rightarrow x^2 \rightarrow x^4 \rightarrow x^8 \rightarrow x^{12} \rightarrow x^{24} \rightarrow x^{25} \rightarrow x^{50} \rightarrow x^{100} \rightarrow x^{200} \rightarrow x^{400} \rightarrow x^{800} \rightarrow x^{1600} \rightarrow x^{2000}$, using **13 multiplications**.

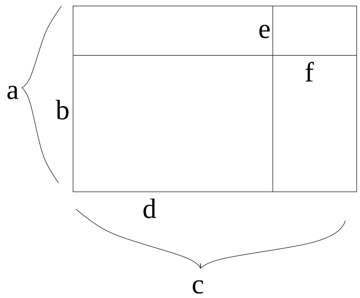
7. All multiplicatively perfect numbers have exactly 4 distinct positive divisors, or 1. So, we must look for numbers that are either

   - 1
   - a product of two distinct primes
   - a cube of a prime

   Numbers satisfying one of these conditions less than 100 are: 1, 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95. There are **33** of these.

8. $168 = 2^3 \cdot 3 \cdot 7$. There are only 2 combinations of these whose sums allow indistinguishability of the ages. If there are 27 trees, 2, 4, 21 and 1, 12, 14 years are possible. If there are 21 trees, 2, 7, 12 and 3, 4, 14 are possible. So, the possible ages of the oldest daughter are **12, 14, 21**.

9. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2 + f^2} = 15$ (since this is $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2}$). Also, note that $ac + bd - ad - bc = (a - b)(c - d) = ef$. So, solving for e and f, we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2 + 4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4}12 = 9$. Therefore, $\boldsymbol{ef = 9 * 12 = 108}$.



10. $a = 1$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 2$, which is prime. $a = 2$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 5$, which is also prime. Here is a table for a's and values of x that show they do not work:

a	x	$a^4 + x^2$
3	10	10009
4	1	17
5	2	41
6	1	37
7	20	160049

So, let us consider $a = 8$ - i.e. the sum $x^4 + 64$. This is the same as $(x^2 + 8)^2 - 16x^2 = (x^2 + 4x + 8)(x^2 - 4x + 8)$ by the difference of squares. This is clearly not prime for any integer x. So, the answer is $\boldsymbol{a = 8}$.