General Solutions
Rice Mathematics Tournament 2000

1. Since $d = a - c$, substitute into the equation $b = 2c + d$ to get $b = 2c + a - c = a + c$. Also, substitute into $2c = d + a - 1$ to get $2c = a - c + a - 1$, or $3c = 2a - 1$. Now, since $b = a + c$, we can substitute into $a = 2b + c$ to get $a = 2a + 2c + c$, or $a = -3c$. Since we know $3c = 2a - 1$ from above, we substitute in to get $3c = -6c - 1$, or $c = -\frac{1}{9}$. Thus, we find that $a = \frac{1}{3}$, $d = \frac{4}{9}$, and $\boldsymbol{b = \frac{2}{9}}$.

2. Let x be the temperature we are looking for, so $x = \frac{9}{5} + 32$. So, $-\frac{4}{5}x = 32$, so $x = -\frac{5}{4} \cdot 32 =$**$-$40**.

3. Let x be the length of Henry's shadow in feet. Using similar triangles, we find that $\frac{5.5}{12} = \frac{x}{5}$, so $x = \frac{5.5}{12} \cdot 5 = \frac{11}{24} \cdot 5 = \mathbf{\frac{55}{24}}$.

4. Let x be the number of students and y be the number of non-students. We then have the equations $x + y = 3000$ and $10x + 15y = 36250$. Substituting, we find that $10(3000 - y) + 15y = 36250$, or $30000 - 10y + 15y = 36250$, so $30000 + 5y = 36250$. So, $5y = 6250$, and $y = 1250$, so $x =$**1750**.

5. The total number of degrees in an octagon is $(8 - 2) \cdot 180 = 1080$. Since the degrees are evenly distributed among the angles, the measure of one interior angle is $\frac{1080}{8} =$**135**$^\circ$.

6. Pick any card first, then pick the other face-down card.
   (a) $\mathbf{\frac{1}{3}}$
   (b) $\mathbf{\frac{2}{3}}$

7. $\sqrt{19992000} \approx 4471.241$, and $[4471.241] =$**4471**.

8. The time that Bobo can juggle is the number of cows times seconds. So, we get the following table:

cows started juggling	1	2	3	4	5	6	7	8	9	10	11
total time	64	110	141	160	165	162	154	144	126	130	132
cows started juggling	12	13	14	15	16	17	18	19	20	21	22
total time	132	130	126	120	112	102	90	76	60	42	22

   Thus, we see that the maximum occurs with **5 cows**, and the total time is 165 seconds $=$ **$2\frac{3}{4}$ minutes**.

9. Let p be the price of HMMT. So $p = k \cdot \frac{x}{y}$, where k is a constant to be determined. We know that when $x = 8$ and $y = 4$ that $p = 12$, so, solving for k, we find that $k = 6$. So, when $x = 4$ and $y = 8$, we find that $p = 6 \cdot \frac{4}{8} =$**3**.

10. On the 12 foot sides, he needs 7 posts, and on the 20 foot sides, he needs 9 posts, so he needs $7 + 9 + 7 + 9 = 32$ total posts. Let x be the number of normal fenceposts and y be the number of strong fenceposts, so $x + y = 32$. To spend \$70, we have the equation $2x + 3y = 70$. Substituting, we find that $2(32 - y) + 3y = 70$, so $64 - 2y + 3y = 70$, and $64 + y = 70$, so $y =$**6**.

11. Substituting, we find that $\frac{3+n}{3-n} = 3$, so $3 + n = 9 - 3n$, thus $4n = 6$. So, $n = \mathbf{\frac{3}{2}}$.

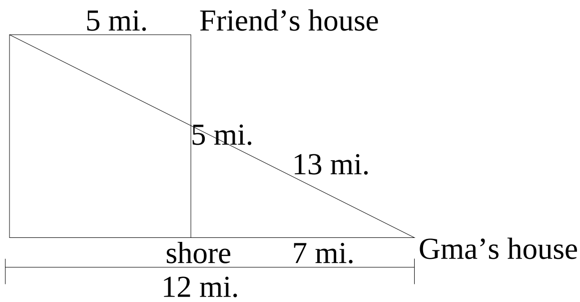
12. Yes - one possible path is $4 \to 1 \to 2 \to 4 \to 5 \to 2 \to 3 \to 5$, so the difference between the start island and end island is **1**.

13. The number of rearrangements keeping 1 number in its spot and rearranging the other 5 such that none are in the right spot is 44. There are 6 numbers to fix, this gives us an answer of $44 \cdot 6 =$**264**.

14. Feburary 26, 2000 is a Saturday. April 24, 2000 is $23 \cdot 365 + 1 + 1 + 1 + 1 + 1 + 1 = 8401$ days away from April 24, 1977 (including the leap years). So, Feburary 26, 2000 is $8401 - 3 - 31 - 24 = 8343$ days after April 24, 1977. Now, $\frac{8343}{7} = 1191$ with a remainder of 6. So, 6 days before Saturday is **Sunday**.

15. Notice that $3^6 = 729$ while $5^4 = 625$, and since $\ln 5 > \ln 3$, it follows that $5^4 \ln 3 < 3^6 \ln 5$, so $5^3 * 5 \ln 3 < 3^5 * 3 \ln 5$, and so by laws of logarithms, $5^3 \ln 3^5 < 3^5 \ln 5^3$. Again applying laws of logarithms, it follows that $\ln (3^5)^{(5^3)} < \ln (5^3)^{(3^5)}$. So, since $\ln x$ is an increasing function, it follows that $(3^5)^{(5^3)} < (5^3)^{(3^5)}$, and so it follows that $\mathbf{(5^3)^{(3^5)}}$ **is greater**. One can also use the laws of exponents to reduce the values to $3^{(5^4)}$ and $5^{(3^6)}$. The second is clearly larger.

16. Expressing the total time Joe has biked in hours leads to the equation $\frac{x}{20} + \frac{x}{20} + \frac{x+2}{14} = 1$. So, $x = 5$. Thus, we can construct the diagram below, and find the total time that it takes to get back: $13 * \frac{1}{78} = \frac{1}{6}$ hours, or **10 minutes**.



17. There are 7! ways to arrange those letters. However, for every distinct arrangement, there are $2! * 2! = 4$ total arrangements of the 2 T's and 2 N's. Therefore, the total number of distinct ways to arrange the letters is $\mathbf{\frac{7!}{2!2!} = 1260}$.

18. The only digits possible are 4, 6, 8, and 9. The only groups of numbers allowed keeping the average at 5 are 8444 are 6464. There are 4 ways to arrange 8444 and 6 ways to arrange 6464 so there are only **10 combinations** to try.

19. Let y be the number of coins in the chest. From the problem, we know that $y \equiv 5 \pmod{11}$, $y \equiv 3 \pmod{10}$, and $y \equiv 0 \pmod{9}$. Combining these gives us that $y \equiv 423 \pmod{990}$, so the answer is **423**.

20. The area of the big circle is $\left(\frac{5}{2}\right)^2 \pi = \frac{25}{4}\pi$. The area of the circle with diameter AB is π, and the area of the circle with diameter BC is $\frac{9}{4}\pi$. Thus, the percentage of the big circle that is shaded is $\frac{\frac{25}{4}\pi - \pi - \frac{9}{4}\pi}{\frac{25}{4}\pi} = \frac{25-4-9}{25} = \mathbf{\frac{12}{25} = 48\%}$.

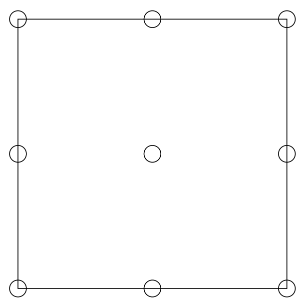
21. It is clear that we can split the figure into 12 equilateral triangles, all of which have side length s. So, since the area of one of these triangles is $\frac{s^2\sqrt{3}}{4}$, the total area is $\mathbf{3s^2\sqrt{3}}$.

22. $A_{iso} = \frac{A_{eq}}{2}$ so $\Delta A = A_{eq} - A_{iso} = \frac{1}{2}A_{eq}$. Also, $A_{eq} = \frac{4^2 \cdot \sqrt{3}}{4} = 4\sqrt{3}$, so $\Delta A = 2\sqrt{3}$. Thus, the area is $\mathbf{2\sqrt{3}}$.

23. Notice that $7^{2k+1} \equiv 3 \pmod 4$ for $k \in \mathbf{Z}$ (k is an integer). Also, $7^1 \equiv 7 \pmod{100}$, $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, $7^4 \equiv 1 \pmod{100}$, with the cycle repeating afterwards. So, clearly $7^{7^{7^7}} \equiv 3 \pmod 4$, and since the cycle has period $4 \pmod{100}$, we can conclude that $\mathbf{7^{7^{7^7}} \equiv 43 \pmod{100}}$.

24. If there are no empty boxes, there are 126 ways of distributing the identical candy. If there is one empty box, there are 84 ways of distributing the candy and 5 ways of choosing the empty box, so there are $84 \cdot 5 = 420$ ways. If there are two empty boxes, there are 36 ways of distributing the candy and 6 combinations of assigning the empty boxes so that they are not adjacent, so there are $36 \cdot 6 = 216$ ways. If there are three empty boxes, there are 9 ways of distributing the candy, and only 1 way of arranging the empty boxes. Having four or five empty boxes is impossible, since some two would have to be adjacent. So, the total number of ways is $126 + 420 + 216 + 9 = \mathbf{771}$.

25. We can place 9 points as shown, all at least $\frac{1}{2}$ unit apart, but the next point must be less than $\frac{1}{2}$, so **10** points must be placed. There is no arrangement of 10 points with distance at least $\frac{1}{2}$. The proof of this is a simple application of the Pigeonhole Principle.



26. Janet is at $(5, -5)$ and Tim is at $(7, 4)$. They are $\mathbf{\sqrt{85}}$ miles apart.