Guts
HMMT 2000

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Problem Gu1 [4]

The sum of 3 real numbers is known to be zero. If the sum of their cubes is π^e, what is their product equal to?

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Problem Gu2 [5]

If $X = 1 + x + x^2 + x^3 + ...$ and $Y = 1 + y + y^2 + y^3 + ...$, what is $1 + xy + x^2y^2 + x^3y^3 + ...$ in terms of X and Y only?

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Problem Gu3 [±7]

Using 3 colors, red, blue and yellow, how many different ways can you color a cube (modulo rigid rotations)?

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Problem Gu4 [5]

Let ABC be a triangle and H be its orthocentre. If it is given that B is $(0,0)$, C is $(1,2)$ and H is $(5,0)$, find A.

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Problem Gu5 [3]

Find all natural numbers n such that n equals the cube of the sum of its digits.

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Problem Gu6 [±10]

If integers m, n, k satisfy $m^2 + n^2 + 1 = kmn$, what values can k have?

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Problem Gu7 [7]

Suppose you are give a fair coin and a sheet of paper with the polynomial x^m written on it. Now for each toss of the coin, if heads show up, you much erase the polynomial x^r (where r is going to change with time – initially it is m) written on the paper and replace it by x^{r-1}. If tails show up, replace it by x^{r+1}. What is the expected value of the polynomial I get after m such tosses? (Note: this is a different concept from the most probable value)

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Problem Gu8 [±4]

Johny's father tells him : "I am twice as old as you will be seven years from the time I was thrice as old as you were". What is Johny's age?

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Problem Gu9 [6]

A cubic polynomial f satisfies $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 4$. What is $f(5)$?

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Problem Gu10 [7]

What is the total surface area of an ice cream cone, radius R, height H, with a spherical scoop of ice cream of radius r on top? (Given $R < r$)

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Problem Gu11 [6]

Let M be the maximum possible value of $x_1x_2 + x_2x_3 + ... + x_5x_1$ where $x1, x2, ..., x5$ is a permutation of $(1, 2, 3, 4, 5)$ and let N be the number of permutations for which this maximum is attained. Evaluate $M + N$.

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Problem Gu12 [9]

Calculate the number of ways of choosing 4 numbers from the set $\{1, 2, ..., 11\}$ such that at least 2 of the numbers are consecutive.

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Problem Gu13 [±4]

Determine the remainder when $(x^4 - 1)(x^2 - 1)$ is divided by $1 + x + x^2$.

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Problem Gu14 [7]

$ABCD$ is a cyclic quadrilateral inscribed in a circle of radius 5, with $AB = 6, BC = 7, CD = 8$. Find AD.

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Problem Gu15 [8]

Find the number of ways of filling a 8×8 grid with 0's and X's so that the number of 0's in each row and each column is odd.

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Problem Gu16 [5]

Solve for real x, y:

$$\begin{aligned} x + y &= 2 \\ x^5 + y^5 &= 82 \end{aligned}$$

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Problem Gu17 [5]

Find the highest power of 3 dividing $\binom{666}{333}$

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Problem Gu18 [±5]

What is the value of $\sum_{n=1}^{\infty} (\tan^{-1} \sqrt{n} - \tan^{-1} \sqrt{n+1})$?

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Problem Gu19 [3]

Define $a * b = \frac{a-b}{1-ab}$. What is $(1 * (2 * (3 * ...(n * (n+1))...)))$?

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Problem Gu20 [6]

What is the minimum possible perimeter of a triangle two of whose sides are along the x-and y-axes and such that the third contains the point (1,2)?

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Problem Gu21 [8]

How many ways can you color a necklace of 7 beads with 4 colors so that no two adjacent beads have the same color?

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Problem Gu22 [6]

Find the smallest n such that 2^{2000} divides $n!$

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Problem Gu23 [5]

How many 7-digit numbers with distinct digits can be made that are divisible by 3?

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Problem Gu24 [± 3]

At least how many moves must a knight make to get from one corner of a chessboard to the opposite corner?

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Problem Gu25 [4]

Find the next number in the sequence 131, 111311, 311321, 1321131211,

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Problem Gu26 [5]

What are the last 3 digits of $1! + 2! + ... + 100!$

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Problem Gu27 [± 6]

What is the smallest number that can be written as a sum of 2 squares in 3 ways?

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Problem Gu28 [8]

What is the smallest possible volume to surface ratio of a solid cone with height = 1 unit?

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Problem Gu29 [± 9]

What is the value of $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}}$

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Problem Gu30 [7]

$ABCD$ is a unit square. If $\angle PAC = \angle PCD$, find the length BP.

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Problem Gu31 [10]

Given collinear points A, B, C such that $AB = BC$. How can you construct a point D on AB such that $AD = 2DB$, using only a straightedge? (You are not allowed to measure distances)

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Problem Gu32 [7]

How many (nondegenerate) tetrahedrons can be formed from the vertices of an n-dimensional hypercube?

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Problem Gu33 [±5]

Characterise all numbers that cannot be written as a sum of 1 or more consecutive odd numbers.

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Problem Gu34 [±6]

What is the largest n such that $n! + 1$ is a square?

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Problem Gu35 [4]

If $1 + 2x + 3x^2 + ... = 9$, find x.

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Problem Gu36 [6]

If, in a triangle of sides a, b, c, the incircle has radius $\frac{b+c-a}{2}$, what is the magnitude of angle A?

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Problem Gu37 [9]

A cone with semivertical angle 30° is half filled with water. What is the angle it must be tilted by so that water starts spilling?

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Problem Gu38 [4]

What is the largest number you can write with three 3's and three 8's, using only symbols +,-,/,× and exponentiation?

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Problem Gu39 [±8]

If $r = 1/3$, what is the value, rounded to 100 decimal digits, of

$$\sum_{n=0}^{7} \frac{2^n}{1 + x^{2^n}}$$

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Problem Gu40 [±10]

Let $\phi(n)$ denote the number of positive integers less than equal to n and relatively prime to n. find all natural numbers n and primes p such that $\phi(n) = \phi(np)$.

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Problem Gu41 [7]

A observes a building of height h at an angle of inclination α from a point on the ground. After walking a distance a toward it, the angle is now 2α, and walking a further distance b causes to increase to 3α. Find h in terms of a and b.

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Problem Gu42 [4]

A $n \times n$ magic square contains numbers from 1 to n^2 such that the sum of every row and every column is the same. What is this sum?

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Problem Gu43 [6]

Box A contains 3 black and 4 blue marbles. Box B has 7 black and 1 blue, whereas Box C has 2 black, 3 blue and 1 green marble. I close my eyes and pick two marbles from 2 different boxes. If it turns out that I get 1 black and 1 blue marble, what is the probability that the black marble is from box A and the blue one is from C?

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Problem Gu44 [6]

A function $f : Z \to Z$ satisfies

$$\begin{aligned} f(x+4) - f(x) &= 8x + 20 \\ f(x^2 - 1) &= (f(x) - x)^2 + x^2 - 2 \end{aligned} \qquad (1)$$

Find $f(0)$ and $f(1)$.

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Problem Gu45 [7]

Find all positive integers x for which there exists a positive integer y such that $\binom{x}{y} = 1999000$

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Problem Gu46 [6]

For what integer values of n is $1 + n + n^2/2 + ... + n^n/n!$ an integer?

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Problem Gu47

Find an $n < 100$ such that $n \cdot 2^n - 1$ is prime. Score will be $n - 5$ for correct n, $5 - n$ for incorrect n (0 points for answer < 5).

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