Oral Event
HMMT 2000

1. [25] Find all integer solutions to $m^2 = n^6 + 1$.

2. [30] How many positive solutions are there to $x^{10} + 7x^9 + 14x^8 + 1729x^7 - 1379x^6 = 0$? How many positive integer solutions ?

3. [35] Suppose the positive integers a, b, c satisfy $a^n + b^n = c^n$, where n is a positive integer greater than 1. Prove that $a, b, c > n$. (Note: Fermat's Last Theorem may *not* be used)

4. [40] On an $n \times n$ chessboard, numbers are written on each square so that the number in a square is the average of the numbers on the adjacent squares. Show that all the numbers are the same.

5. [45] Show that it is impossible to find a triangle in the plane with all integer coordinates such that the lengths of the sides are all odd.

6. [45] Prove that every multiple of 3 can be written as a sum of four cubes (positive or negative).

7. [45] A regular tetrahedron of volume 1 is filled with water of total volume 7/16. Is it possible that the center of the tetrahedron lies on the surface of the water? How about in a cube of volume 1?

8. [55] f is a polynomial of degree n with integer coefficients and $f(x) = x^2 + 1$ for $x = 1, 2, ..., n$. What are the possible values for $f(0)$?

9. [60] Let $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ and $\vec{v_5}$ be vectors in three dimensions. Show that for some i, j in $1, 2, 3, 4, 5$, $\vec{v_i} \cdot \vec{v_j} \geq 0$.

10. [75] 23 frat brothers are sitting in a circle. One, call him Alex, starts with a gallon of water. On the first turn, Alex gives each person in the circle some rational fraction of his water. On each subsequent turn, every person with water uses the same scheme as Alex did to distribute his water, but in relation to themselves. For instance, suppose Alex gave 1/2 and 1/6 of his water to his left and right neighbors respectively on the first turn and kept 1/3 for himself. On each subsequent turn everyone gives 1/2 and 1/6 of the water they started the turn with to their left and right neighbours (resp.) and keep the final third for themselves. After 23 turns, Alex again has a gallon of water. What possibilities are there for the scheme he used in the first turn? (Note: you may find it useful to know that $1 + x + x^2 + ... + x^{22}$ has no polynomial factors with rational coefficients.)