Oral Round Solutions
HMMT 2000

1. The equation involves only even powers, so we may assume m, n are positive (negative solutions may be generated by switching signs). $n^6 \leq n^6 + 1 \leq n^6 + 2n^3 + 1$, so $n^6 + 1$ lies between two squares, with strict inequality unless $n = 0$. So the only solutions are $m = \pm 1, n = 0$.

2. Dividing by x^6, we get the equation $x^4 + 7x^3 + 14x^2 + 1729x - 1379 = 0$. This is a monotonically increasing function of x on $[0, \infty)$. From $f(0) < 0$ and $f(x) \to \infty$ as $x \to \infty$ we see that there's only one positive solution. In fact, $f(1) > 0$, so there's no positive integer solution.

3. Assume w.lo.g that $c > b \geq a$. Then $c^n - b^n = (c - b)(c^{n-1} + ... + b^{n-1}) > 1(nb^{n-1}) \geq na^{n-1}$. So $a^n > na^{n-1}$, hence $a > n$.

4. Let m be the smallest number, written on some square S. Then clearly all of the adjacent squares to S must have m on them (they can't have anything smaller, since m is the smallest, and can't have anything larger since m is the average of those adjacent numbers). Since this applies to every square on which m is written, all of the squares must have m on them. Otherwise there will be some "boundary" square with m on it, which will not have m written on one of its neighbours, a contradiction.

5. Suppose there is such a triangle with vertices at $(m_1, n_1), (m_2, n_2), (m_3, n_3)$. Then we are given $\sqrt{(m_1 - m_2)^2 + (n_1 - n_2)^2}$ is an odd integer, so that $(m_1 - m_2)^2 + (n_1 - n_2)^2$ is 1 (mod 4). Hence exactly one of $m_1 - m_2$ and $n_1 - n_2$ must be odd and the other must be even. Suppose w.lo.g $m_1 - m_2$ is odd and $n_1 - n_2$ is even. Similarly, exactly one of $m_1 - m_3$ and $n_1 - n_3$ must be odd and the other must be even. If $m_1 - m_3$ is odd and $n_1 - n_3$ even, then $m_2 - m_3$ and $n_2 - n_3$ are both even, which is a contradiction sice we want $(m_2 - m_3)^2 + (n_2 - n_3)^2$ to be 1 (mod 4). Similarly if $m_1 - m_3$ is even and $n_1 - n_3$ is odd then $m_2 - m_3$ and $n_2 - n_3$ are both odd, and $(m_2 - m_3)^2 + (n_2 - n_3)^2$ is 2 (mod 4), a contradiction.

   Alternate solution: By Pick's theorem ($A = I + \frac{1}{2}B - 1$), the area is an integer or a half-integer. But by Hero's formula, the area is $\frac{1}{4}\sqrt{\text{product of 4 odd integers}}$. this is a contradiction.

6. Any multiple of 3 can be written as $6k$ or $6k + 15$. Now, $6k = (k + 1)^3 + (k - 1)^3 + (-k)^3 + (-k)^3$. $6k + 15 = (k + 2)^3 + (-k + 2)^3 + (2k - 1)^3 + (-2k)^3$.

7. If the tetrahedron is oriented so that two opposite edges of the tetrahedron are parallel to the ground, it is clear from symmetry that the plane parallel to the ground and passing through the center of the tetrahedron splits it into equal spaces of volume 1/2. Let's call this position 1. If, instead, it is oriented with the point down and the top face parallel to the ground, the plane through the center splits it into regions of volume 27/64 and 37/64. To see this, note that the center of a regular tetrahedron is 3/4 of the way down the altitude from each vertex. Hence the tetrahedral section has volume $(3/4)^3 = 27/64$. Let's call this position position 2. If the tetrahedron is turned continuously from position 1 to position 2, because $27/64 < 7/16 < 1/2$, it must pass through an orientation s.t. the fraction of volume below the center is 7/16.

   The cube has 180° rotational symmetry, so any plane through the center splits it into sections of equal volume. Hence it is impossible to get a volume of 7/16.

8. Let f be one such polynomial, for example, the obvious one: $f(x) = x^2 + 1$. Then all such polynomials must be of the form $f(x)+g(x)$ where $g(x) = 0$ at $x = 1, 2, ..., n$. $g(x)$ must of the form $C(x-1)(x-2)...(x-n)$ and since $f+g$ has to have integer coefficients, C must be an integer. At 0 this gives us $(f+g)(0) = f(0)+g(0) = 1+C(-1)^n n!$. Since C can be any positive or negative integer, the answer is that f(0) can be any integer congruent to $1 (\text{mod } n!)$, i.e. of the form $1 + Cn!$, C an integer.

9. This is a special case of the problem with $n + 2$ vectors in n dimensions. First it is clear that we can take all the vectors to be of length 1. Then we induct on n. The first case is n=1. Here the statement is that given a, b, c real numbers, the at least one of ab, bc, and ac is nonnegative. Without loss of generality, we can assume that a and b are of the same sign, but then $ab \geq 0$. Now assume the statement is false for the n-dimensional case. Choose some vector, say $\overrightarrow{v_{n+2}}$, and project the other vectors onto the space perpendicular to $\overrightarrow{v_{n+2}}$ to get $\overrightarrow{v_i'} = \overrightarrow{v_i} - (\overrightarrow{v_i} \cdot \overrightarrow{v_{n+2}})\overrightarrow{v_{n+2}}$. This is essentially taking out the space parallel to $\overrightarrow{v_{n+2}}$ and reducing the problem by one dimension. The only thing left to check is that if $\overrightarrow{v_i} \cdot \overrightarrow{v_j} < 0$ for all i,j then $\overrightarrow{v_i'} \cdot \overrightarrow{v_j'} < 0$ for all i,j. This is just a calculation: $\overrightarrow{v_i'} \cdot \overrightarrow{v_j'} = \overrightarrow{v_i} \cdot \overrightarrow{v_j} - 2(\overrightarrow{v_j} \cdot \overrightarrow{v_{n+2}})(\overrightarrow{v_i} \cdot \overrightarrow{v_{n+2}}) + (\overrightarrow{v_{n+2}} \cdot \overrightarrow{v_{n+2}})(\overrightarrow{v_j} \cdot \overrightarrow{v_{n+2}})(\overrightarrow{v_i} \cdot \overrightarrow{v_{n+2}})$ and since $(\overrightarrow{v_{n+2}} \cdot \overrightarrow{v_{n+2}}) = 1$, this is just $\overrightarrow{v_i} \cdot \overrightarrow{v_j} - (\overrightarrow{v_j} \cdot \overrightarrow{v_{n+2}})(\overrightarrow{v_i} \cdot \overrightarrow{v_{n+2}})$. $(\overrightarrow{v_j} \cdot \overrightarrow{v_{n+2}})$ and $(\overrightarrow{v_i} \cdot \overrightarrow{v_{n+2}})$ are both negative by assumption so their product is positive and $\overrightarrow{v_i'} \cdot \overrightarrow{v_j'} < \overrightarrow{v_i} \cdot \overrightarrow{v_j} < 0$.

10. Let ω be a 23rd root of unity, a solution to the equation $x^{23} = 1$. We can encode the state of the game as 23 rational numbers $a_0, ..., a_{22}$ representing the amount of water each person has, with Alex having the frction a_0 of the water and continuing to his right. Then we can look at the polynomial $f(x) = a_{22}x^{22} + ... + a_0$ which still encodes the state of the game. In this schema the original state is the constant 1, and we have at all times the condition $f(1) = 1$ (conservation of water). However, we want a cyclic representation in order to encode the play of the game, so we let the state of the game be $f(\omega) = a_{22}\omega^{22} + ... + a_0$. In this representation, a step in the game can be encoded as multiplication by another polynomial in ω, $g(\omega) = b_{22}\omega^{22} + ... + b_0$ where the b_i are the fraction of water that Alex initially gives to the i-th person on his right. The example in the problem is represented by $\frac{1}{6}\omega + \frac{1}{3} + \frac{1}{2}\omega^{22}$. So we would like to know which g are solutions to $g(\omega)^{23} = 1$. We can take 23rd roots of both sides and find that $g(\omega) = b_{22}\omega^{22} + ... + b_0 = \omega^k$ for some integer k. To simplify this we need a small lemma.

Lemma: If $a_{22}\omega^{22} + ... + a_0$ is some rational linear combination of the powers of ω and is equal to 0, then $a_{22} = a_{21} = ... = a_0$.

Proof: Assume this is false. So there exists some polynomial $A(x) = a_{22}x^{22} + ... + a_0$ with ω as a root that is not a constant multiple of $B(x) = x^{22} + x^{21} + ... + 1$. Now apply Euclid's algorithm to find the G.C.D. of $A(x)$ and $B(x)$ and call it $D(x)$. But $D(x)$ is a rational polynomial that divides $B(x)$, which has no non-trivial rational divisors. Hence $D(x)$ must be either a constant or of degree 22. Since $A(\omega) = B(\omega) = 0$ all the terms in the algorithm have a root at ω and so does $D(x)$. So since 0 is not an acceptable value for a G.C.D., $D(x)$ must have degree 22. Since the degree of each successive term in Euclid's algorithm decreases by at least one, this means that $D(x)$ must be $A(x)$ and hence that $B(x)$ is a constant multiple of $A(x)$. This contradicts the original assumption and the Lemma is proved.

So the Lemma tells us that $b_{22} = b_{21} = ... = b_k - 1 = ... = b_0$. Since we already know that $g(1) = 1$ and hence $b_{22} + b_{21} + ... + b_0 = 1$, and that all the b_i are nonnegative, we get that $b_i = 0$ for $i \neq k$ and $b_k = 1$. Since this is a solution for all k, the possible schemes for Alex to use are just those involving giving all his water to any one other person.