Power Test
Rice Mathematics Tournament 2000

1. A derangement of a string of distinct elements is a rearrangement of the string such that no element appears in its original position. For example, BCA is a derangement of ABC. D_n represents the number of derangements of any string composed of n distinct elements. $D_2 = 1$ and $D_3 = 2$.
   (a) What are D_4 and D_5?
   (b) How many derangments are there of the string ABCDEFG?
   (c) Find a recursive relationship for D_n in terms of the previous two terms (D_{n-1} and D_{n-2}).
   (d) Find a recursive relationship for D_n in terms of only the previous term, D_{n-1}.
2. Find the number of 3-letter "words" that use letters from the 10-letter set $\{A, B, C, ..., J\}$ in which all letters are different and the letters appear in alphabetical order.
3. Assume that a hand of thirteen cards is dealt from a randomized deck of 52 cards. Let A be the probability that the hand contains two aces. Let B be the probability that it contains two aces if you already know it contains at least one ace. Let C be the probability that it contains at least two aces if you already know it contains an ace of hearts. Write down the inequality relationship between A,B and C, i.e. one possibility is $A = B > C$.
4. Find the number of rearrangments of 12345 (including 12345) such that none of the following is true: 1 is in position 5, 2 is in position 1, 3 is in position 2, 4 is in position 4, and 5 is in position 3.
5. Assume that nobody has a birthday on February 29th. How large must a group be so that there is a greater than 50% chance that at least 2 members have the same birthday?
6. Find the number of combinations of length k that use elements from a set of n distinct elements, allowing repetition.
7. Find the number of combinations of length k that use elements from a given set of n distinct elements, allowing repetition and with no missing elements. (Obviously, k must be greater than n)
8. An election takes place between two candidates. Candidate A wins by a vote of 1032 to 971. If the votes are counted one at a time and in a random order, determine the probability that the winner was never behind at any point in the counting.
9. Evaluate the sum $\binom{100}{0} + \frac{1}{2}\binom{100}{1} + \frac{1}{3}\binom{100}{2} + \ldots + \frac{1}{101}\binom{100}{100}$.
10. Find the number of distributions of a given set of m identical balls into a given set of n distinct boxes.
11. Find the number of distributions of a given set of m distinct balls into a given set of n distinct boxes if each box must contain a specific number of balls (m_i is the number of balls to be put into box i). Please state the answer with only factorials (not in combinatorial notation).

12. Find the coefficient of X^2Y^3 in each of the following.
    (a) $(X+Y+1)^7$
    (b) $(X^2+Y-1)^7$
13. Find the number of "words" of length m from a set of n letters, if each letter must occur at least once in each word.
14. Find the number of ways to distribute seven distinct balls into three distinct boxes if each box must contain a different number of balls, allowing an empty box.
15. How many ways can a class of 10 students be divided into two groups of 3 and 1 group of 4?
16. Find the number of subsets A of the set of digits $\{0,1,2,3,\ldots,9\}$ such that A contains no two consecutive digits. Hint: Find a better statement of the problem; find a recursive formula, and then attempt to solve the problem for the number of digits given.
17. If we are trying to find the number of words of length m from a given set of n letters, with each letter occuring at least once in each word, let us call the answer $T(m,n)$. This is equivalent to finding the number of distribution of a set of m distinct balls into a set of n distinct boxes, if no boxes can be empty. $T(m,n)$ is the sum of all possible partitions of the balls (i.e. we sum all possible ways of putting the balls into boxes (4 in box 1, 2 in box 2, 1 in box 3 for example)). More precisely, if we call m_i to be the number of balls in box i, then $T(m,n) = \sum_{m_1,m_2,m_3>=1}^{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!...m_n!}$. For example, $T(3,2) = \frac{3!}{1!2!} + \frac{3!}{2!1!} = 3+3 = 6$. Find a recursive pattern for $T(m,n)$ in terms of previous terms (previous meaning a smaller m, a smaller n, or both). Hint: set up a sort of "Pascal's Triangle" for $T(m,n)$. Prove your answer using words.
18. You have an infinite number of 1 cent, 2 cent, and 5 cent stamps. You are trying to post a letter that requires n cents of postage stamps, where $n > 8$. Let $a(n)$ be the number of sequences of stamps that give exactly the required postage of n cents (i.e. order matters). Find $a(n)$ in terms of previous terms of the sequence of a's, using as few previous terms as possible.
19. Suppose we have n lines in a plane in general position, which means that none are parallel to each other and that no three of these lines intersect at a single point. Find the number of regions that these lines divide the plane into...
    (a) in a recursive form.
    (b) in a nonrecursive formula.
20. Find the 2000th positive integer that is not the difference between any two integer squares.