Power Test Solutions
Rice Mathematics Tournament 2000

1. (a) $\boldsymbol{D_4 = 9, D_5 = 44}$
   (b) This is $D_7 =$**1854**
   (c) $\boldsymbol{D_n = (n-1)(D_{n-1} + D_{n-2})}$
   (d) $\boldsymbol{D_n = n \cdot D_{n-1} + (-1)^n}$
2. $10 \cdot 9 \cdot 8$ give the number of all different 3 letter words. One sixth of these are in alphabetical order. $\frac{10 \cdot 9 \cdot 8}{6} = \mathbf{120}$.
3. $\boldsymbol{A < B < C}$. More specifically, $A = .26$; $B = .37$; $C = .56$.
4. This is just derangements of 23541. $D_5 =$**44**.
5. We want $1 - 365 \cdot 364 \cdot 363 \ldots (365 - n + 1)/365^n > 50\%$. Just guestimating gives the answer of **23** people.
6. Since we want to choose k times from n distinct elements, this is equivalent to choosing where to put $n-1$ "dividers" that separate the choices. For example, if we wanted to choose 3 scoops of ice cream from the flavors chocolate, vanilla, strawberry, and coffee, we can represent the choice of 2 vanilla and 1 coffee by (divider),choice,choice,(divider),(divider),cho Notice that the choices of positions for the dividers completely determines which elements we choose. Therefore, we have $n+k-1$ spaces to fill with $n-1$ dividers, so the number of ways of doing this is $\boldsymbol{\binom{n+k-1}{n-1}}$ or $\boldsymbol{\binom{n+k-1}{k}}$
7. We can use the same method as the above argument, except that we know every element occurs at least once. So, this is the same as choosing $k-n$ times from the n distinct objects, so the answer is $\binom{k-n-n+1}{n-1}$, or $\boldsymbol{\binom{k-1}{n-1}}$ or $\boldsymbol{\binom{k-1}{k-n}}$.
8. FIX????????? $1 - \frac{971}{1033} = \mathbf{\frac{62}{1033}}$
9. Let S be the sum desired. Then $101 \cdot S = \frac{101}{1}\binom{100}{0} + \frac{101}{2}\binom{100}{1} + \frac{101}{3}\binom{100}{2} + \ldots + \frac{101}{101}\binom{100}{100}$. Now, consider a general term in this expression - i.e. $\frac{101}{i+1}\binom{100}{i}$. This is equal to $\frac{101}{i+1} \cdot \frac{100!}{(100-i)!i!} = \frac{101 \cdot 100!}{(101-(i+1))!i! \cdot (i+1)} = \frac{101!}{(101-(i+1))!i+1!} = \binom{101}{i+1}$. So, we can simplify the terms to get $101 \cdot S = \binom{101}{1} + \binom{101}{2} + \binom{101}{3} + \ldots + \binom{101}{100} + \binom{101}{101}$. Thus, $101 \cdot S = 2^{101} - 1$, so $\boldsymbol{S = \frac{2^{101}-1}{101}}$.
10. $\boldsymbol{\binom{n+m-1}{m}}$ or $\boldsymbol{\binom{n+m-1}{n-1}}$. This can be seen from the fact that each distribution can be described by a combination of letters, where each letter represents a different box. The number of times each letter occurs in the combination is determined by the number of balls in the corresponding box.
11. $\boldsymbol{\frac{m!}{[m_1!m_2!m_3!...m_n!]}}$. This is also equal to $\binom{m}{m_1}\binom{m-m_1}{m_2}\binom{m-m_1-m_2}{m_3}\ldots$
12. This is very similiar to 11. A)$\frac{7!}{2!3!2!}$ =**210** B)$-7!/(3!3!)$ =**-140**
13. This is the sum of all the distribution numbers in which the numbers $m_1 \ldots m_n$ run through all possible sequences of n positive integers adding up to m: $\boldsymbol{\sum_{m_i >= 1}^{m_1+\ldots m_n = m} \frac{m!}{m_1!m_2!\ldots m_n!}}$

14. The boxes can be in any order, so we have a factor of 6. Seven can be partitioned into three distinct parts as 0,1,6; 0,2,5; 0,3,4; or 1,2,4. So, the answer is $6\left[\frac{7!}{0!1!6!}+\frac{7!}{0!2!5!}+\frac{7!}{0!3!4!}+\frac{7!}{1!2!4!}\right]=$ **1008**.

15. This is partitioning 10 into 3 partitions, 2 of 1 type and one of the other. Thus, the answer is $\frac{10!}{3!3!4!2!}$ =**2100**.

16. This is equivalent to finding the number of sequences of length 10 composed of 0's and 1's. (0 in a spot corresponds to that spot's number (0-9) not being in the subset.) However, we can't have two consecutive 1's. If we try to generalize, let $0 = A$, $1 = B$ and we are doing an n-letter "word" instead of ten. Set w_n = number of n-letter words (satisfying the conditions); set a_n = number of words counted by w_n that begin with A; set b_n = number of words counted by w_n that begin with B. $w_n = a_n + b_n$. $a_n = b_{n-1}$. $b_n = w_{n-1}$. Combining these we get the recursive relationship $w_n = w_{n-1} + w_{n-2}$. Then we can build up to find that $\boldsymbol{w_{10} = 144}$.

17. $\boldsymbol{T(m,n) = n(T(m-1,n-1) + T(m-1,n))}$ for $1 < n < m$. To prove this, look at $T(5,3)$ and think of it as the number of 5-letter words from $\{A,B,C\}$ with no missing letters. There are 3 choices for the first letter. After this, the remaining four letters must be filled in, and the first letter (call it X) does not have to be used again. There are two cases:

   - If X does not occur again, then the word can be completed in $T(4,2)$ ways.
   - If X does occur again, then the number of ways to complete the word is $T(4,3)$.

   As we have n choices for the letter X (first letter), we get that $T(5,3) = 3 \cdot (T(4,2) + T(4,3))$, or the above, in general.

18. $\boldsymbol{a_n = a_{n-1} + a_{n-2} + a_{n-5}}$. Every way to make n cents ends in either a 1 cent, 2 cents, or 5 cents (since order matters, there is a distinct last stamp). There are a_{n-1} ways to make n cents ending in a 1 cent stamp, since this is the number of ways to make $n-1$ cents. Similarly, there are a_{n-2} ways to make n cents ending in a 2 cents stamp, and a_{n-5} ways to make n cents ending in a 5 cents stamp. Since every way to make n cents ends in one of these stamps, and there is no overlap, the total number of ways to make n cents is the sum of these, or $a_{n-1} + a_{n-2} + a_{n-5}$.

19. (a) $\boldsymbol{r_n = r_{n-1} + n}$

    (b) $\boldsymbol{\binom{n+1}{2} + 1}$

20. First, we will show that a positive integer x is not a difference of squares if and only if $x \equiv 2 \pmod 4$.

    First, suppose $x = a^2 - b^2$ for some integers a, b, so $x = (a-b)(a+b)$. Now, if $a - b = 1$, then $a + b = 2n + 1$ for some integer n, so $x = (a-b)(a+b) = 2n+1$, and x is odd.

    Conversely, if x is odd, then $x = 2n+1$ for some integer n, so $x = 1(2n+1) = (n+1-n)(n+1+n) = (n+1)^2 - n^2$.

    Now, if $a - b = 2$, then $a + b = 2n + 2$ for some integer n, so $x = (a-b)(a+b) = 2(2n+2) = 4n+4$

    Conversely, if $x \equiv 0 \pmod 4$, then $x = 4n+4$ for some integer n, so $x = (n+2)^2 - n^2$.

    So, for all other cases, $a - b \geq 2$. If $a - b$ is even, then we know $a - b = 2n$ for some integer n, and $a + b = 2n + 2m$ for some integer m, so $x = (a-b)(a+b) = 2n \cdot (2n+2m) = 4nm + 4n^2$, so $x \equiv 0 \pmod 4$, and we have already covered this case.

If $a-b$ is odd, then we know $a-b=2n+1$ for some integer n, and $a+b=2n+2m+1$, where m is some integer, so $x=(a-b)(a+b)=(2n+1)\cdot(2n+2m+1)=4(n^2+nm+n)+2n+2m+1\equiv 1 \pmod 4$, which is a case that we have already covered.

Therefore, x is not a difference of two squares if and only if $x\equiv 2 \pmod 4$. So, the 2000th number is $4\cdot 1999+2=$**7998**.