Team Test Solutions
Rice Mathematics Tournament 2000

1. .6445 rounds to .645 to .65 to .7. Otherwise .6444... rounds to .644. So the smallest number is **.6445**.

2. Let c =price, p=purity, d =diameter, h=depth of gold mine, k_i=constant. We are given $c = k_1 p^2 d^3$, $p = k_2 \frac{c^2}{h^2}$, and $d = k_3 \sqrt[3]{c}h$. So, $c = k_1 k_2^2 \frac{c^4}{h^4} k_3^3 ch^3 = k_4 c^5 \frac{1}{h}$. Thus, $k_4 c^4 = h$, and $c = k_5 h^{\frac{1}{4}}$. Thus, p varies as $\boldsymbol{h^{\frac{1}{4}}}$.

3. The sum of the numbers from 700 to 799 is $\frac{799\cdot 800}{2} - \frac{699\cdot 700}{2} = 74950$. The sum of the numbers from 70 to 79 is $\frac{79\cdot 80}{2} - \frac{69\cdot 70}{2} = 745$. So, all numbers that end from 70 to 79 (excluding those starting with 7, since we counted those from 700 to 799) is $745\cdot 9 + 10(100 + 200 + \ldots + 600 + 800 + 900) = 44705$. The sum of all numbers ending in 7 is $9(7+17+27+37+47+57+67+87+97)+9(100+200+\ldots+600+800+900) = 38187$. So, the total sum of numbers containing a 7 is $74950 + 44705 + 38186 =$ **157842**.

4. **738,826**. This can be arrived at by stepping down, starting with finding how many combinations are there that begin with a letter other than V or W, and so forth. The answer is $\frac{8\cdot 9!}{2\cdot 2} + \frac{4\cdot 7!}{2} + 4\cdot 6! + 4\cdot 4! + 3! + 2! + 2! =$ **738826**.

5. $0 = \cos(\alpha+\beta) + \sin(\alpha-\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta + \sin\alpha\cdot\cos\beta - \sin\beta\cos\alpha = (\cos\alpha + \sin\alpha)\cdot(\cos\beta - \sin\beta)$. So $\cos\alpha + \sin\alpha = 0$ or $\cos\beta - \sin\beta = 0$. Then $\tan\alpha = -1$ or $\tan\beta = 1$. Since $\tan\beta$ is given as $\frac{1}{2000}$, $\tan\alpha =$**−1**.

6. Since $\alpha^3 - \alpha - 1 = 0$, then $\alpha^{10} = \alpha^8 + \alpha^7$. So, we can reduce our expression to $3\alpha^8 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $3\alpha^8 - 3\alpha^6 - 3\alpha^5 = 0$, so our expression reduces to $4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $4\alpha^4 - 4\alpha^2 - 4\alpha^0$, so our expression reduces to $2\alpha^3 - 2\alpha - 17$. Now, $2\alpha^3 - 2\alpha - 2 = 0$, so our expression reduces to **-15**, which is our answer.

7. Another 4-digit number that satisfies this property is **9801**, since 9801=9*1089.

8. If she has a silver dollar, then she would have too many other coins, as 0 half dollars, 2 quarters, 3 dimes, etc. would be greater than the total. So she has no silver dollars, and at least one of every other denomination. Continuing, it turns out the only feasible solution is 0 silver dollars, 1 half dollar, 2 quarters, 3 dimes, 4 nickels, 8 pennies, for a total of **18** coins.

9. It suffices to consider $x \geq 1$, since $4(-x)^4 + 1 = 4(x)^4 + 1$, and $4(0) + 1 = 1$ is not prime. So, $4x^4 + 1 = (4x^4 + 4x^2 + 1) - 4x^2 = (2x^2+1)^2 - (2x)^2 = (2x^2 + 1 - 2x)(2x^2 + 1 + 2x)$. For integers x, both $2x^2 - 2x + 1$ and $2x^2 + 2x + 1$ are integers, so this factors $4x^4 + 1$ unless $2x^2 - 2x + 1 = \pm 1$ or $2x^2 + 2x + 1 = \pm 1$. Since $x > 0$, then $2x^2 + 2x + 1 > 1$, so we must have $2x^2 - 2x + 1 = \pm 1$. $2x^2 - 2x + 1 = -1$ is absurd ($4x^4 + 1, 2x^2 + 2x + 1 > 0$, so $2x^2 - 2x + 1 = \frac{4x^4+1}{2x^2+2x+1} > 0$), so we solve $2x^2 - 2x + 1 = 1$, or $2x^2 - 2x = 0$, so $x(x-1) = 0$, and $x = 0$ or $x = 1$. We have already rejected $x = 0$, so the only case left is $x = 1$, or $4(1)^4 + 1 =$ **5**.

10. The second hand crosses the minute hand 59 times an hour. The second hand crosses the hour hand 60 times an hour, except for 2 of the hours, due to the movement of the hour hand. The minute hand and the hour hand cross 22 times total, because the hour hand

completes 2 rotations in a day, and the minute hand completes 24. The second, hour, and minute hand all coincide only at noon and midnight, but we've counted each of these 12:00's 3 times instead of once. Therefore, the answer is $59 \cdot 60 + 60 \cdot 24 - 2 + 22 - 2 \cdot 2$, giving us **2872** crossings.

11. $f(x)$ is either $\mathbf{0}$ or something of the form $\pm \boldsymbol{x^m}$, where $m \geq 0$.

12. A's position is $(a - V_a t, 0)$ and P's position is $(0, b - V_b t)$. So, at time t, the distance between them is $\sqrt{(a - V_a t)^2 + (b - V_b t)^2}$. Notice that this distance is the same as the distance between the point (a, b) and the line $(V_a t, V_b t)$, which is the same as the line $V_b x - V_a y = 0$. The distance from a line $Ax + By + C = 0$ and (x_0, y_0) is $\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, so the answer is $\boldsymbol{\frac{|bV_a - aV_b|}{\sqrt{V_a^2 + V_b^2}}}$

13. Given any 4 vertices, there is exactly one intersection of all the diagonals connecting them. So, the answer is $\boldsymbol{\binom{n}{4}}$.

14. x_0 if $n \equiv 0 \pmod 4$,$\frac{1+x_0}{1-x_0}$ if $n \equiv 1 \pmod 4$, $-\frac{1}{x_0}$ if $n \equiv 2 \pmod 4$, $\frac{x_0 - 1}{x_0 + 1}$ if $n \equiv 3 \pmod 4$.

15. Consider a regular n-gon with radius r. Let x be the side length of the n-gon. So, since the central angle is $\frac{2\pi}{n}$ (see diagram below), use the Law of Cosines to find that $x^2 = r^2 + r^2 - 2r * r \cos \frac{2\pi}{n}$, so $x^2 = 2r^2(1 - \cos \frac{2\pi}{n})$. Thus, $x = r\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. So, the total perimeter of the n-gon is $nx = nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. Now, if we take $\lim_{n \to \infty}$ of the perimeter, the result will be $2\pi n$, since the n-gon approaches a cirle, so $\lim_{n \to \infty} nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}} = 2\pi r$, and so $\boldsymbol{\lim_{n \to \infty} nr\sqrt{1 - \cos \frac{2\pi}{n}} = \pi r \sqrt{2}}$.