Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Combinatorics Subject Test — Solutions

1. You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, and all configurations of switches are equally likely. You perform the following operation: for each switch S, if the two switches next to S were initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?

   Solution: $\boxed{1/2^{2002}}$

   There are 2^{2003} equally likely starting configurations. All switches end up ON if and only if switches $1, 3, 5, 7, \ldots, 2003, 2, 4, \ldots, 2002$ — i.e. all 2003 of them — were initially in the same position. This initial position can be ON or OFF, so this situation occurs with probability $2/2^{2003} = 1/2^{2002}$.

2. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

   Solution: $\boxed{36}$

   Since each row has only two squares, it is impossible for two marked squares to be in the same row. Therefore, exactly nine of the ten rows contain marked squares. Consider two cases:

   Case 1: The first or last row is empty. These two cases are symmetrical, so assume without loss of generality that the first row is empty. There are two possibilities for the second row: either the first square is marked, or the second square is marked. Since the third row must contain a marked square, and it cannot be in the same column as the marked square in the second row, the third row is determined by the second. Similarly, all the remaining rows are determined. This leaves two possibilities if the first row is empty. Thus, there are four possibilities if the first or last row is empty.

   Case 2: The empty row is not the first or last. Then, there are two blocks of (one of more) consecutive rows of marked squares. As above, the configuration of the rows in each of the two blocks is determined by the position of the marked square in the first of its rows. That makes $2 \times 2 = 4$ possible configurations. There are eight possibilities for the empty row, making a total of 32 possibilities in this case.

   Together, there are 36 possible configurations of marked squares.

3. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

   Solution: $\boxed{9/13}$

   Consider the situation after two points. Daniel has a 9/25 chance of winning, Scott, 4/25, and there is a 12/25 chance that the players will be tied. In the latter case, we revert to the original situation. In particular, after every two points, either the game returns to the original situation, or one player wins. If it is given that the game lasts $2k$ rounds, then the players must be at par after $2(k-1)$ rounds, and then Daniel wins with probability $(9/25)/(9/25+4/25) = 9/13$. Since this holds for any k, we conclude that Daniel wins the game with probability $9/13$.

4. In a certain country, there are 100 senators, each of whom has 4 aides. These senators and aides serve on various committees. A committee may consist either of 5 senators, of 4 senators and 4 aides, or of 2 senators and 12 aides. Every senator serves on 5 committees, and every aide serves on 3 committees. How many committees are there altogether?

   Solution: $\boxed{160}$

   If each senator gets a point for every committee on which she serves, and every aide gets 1/4 point for every committee on which he serves, then the 100 senators get 500 points altogether, and the 400 aides get 300 points altogether, for a total of 800 points. On the other hand, each committee contributes 5 points, so there must be $800/5 = 160$ committees.

5. We wish to color the integers $1, 2, 3, \ldots, 10$ in red, green, and blue, so that no two numbers a and b, with $a - b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

   Solution: $\boxed{186}$

   The condition is equivalent to never having an odd number and an even number in the same color. We can choose one of the three colors for the odd numbers and distribute the other two colors freely among the 5 even numbers; this can be done in $3 \cdot 2^5 = 96$ ways. We can also choose one color for the even numbers and distribute the other two colors among the 5 odd numbers, again in 96 ways. This gives a total of 192 possibilities. However, we have double-counted the $3 \cdot 2 = 6$ cases where all odd numbers are the same color and all even numbers are the same color, so there are actually $192 - 6 = 186$ possible colorings.

6. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

   Solution: $\boxed{0}$

   Color the chairs red and black in checkerboard fashion, with the center chair black. Then all 18 red chairs are initially occupied. Also notice that adjacent chairs have different colors. It follows that we need 18 black chairs to accommodate the reassignment, but there are only 17 of them. Thus, the answer is 0.

7. You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, \ldots$. Each ball has probability $1/2^n$ of being put into box n. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

   Solution: $\boxed{5/7}$

Notice that the answer is the sum of the probabilities that boxes $1, 2, \ldots$, respectively, contain at least 2 balls, since those events are mutually exclusive. For box n, the probability of having at least 2 balls is

$$3[(1/2^n)^2(1-1/2^n)] + (1/2^n)^3 = 3/2^{2n} - 2/2^{3n} = 3/4^n - 2/8^n.$$

Summing to infinity using the geometric series formula, we get the answer $(3/4)/(1 - 1/4) - (2/8)/(1-1/8)$, which is equal to $5/7$.

8. For any subset $S \subseteq \{1, 2, \ldots, 15\}$, a number n is called an "anchor" for S if n and $n + |S|$ are both members of S, where $|S|$ denotes the number of members of S. Find the average number of anchors over all possible subsets $S \subseteq \{1, 2, \ldots, 15\}$.

   Solution: $\boxed{13/8}$

   We first find the sum of the numbers of anchors of all subsets S; this is equivalent to finding, for each n, the number of sets for which n is an anchor, and then summing over all n. Suppose that n is an anchor for S, and S has k elements. Then $n, n+k \in S \Rightarrow k \geq 2$, and also $n + k \leq 15$, or $k \leq 15 - n$. The remaining $k - 2$ elements of S (other than n and $n+k$) may be freely chosen from the remaining 13 members of $\{1, 2, \ldots, 15\}$, so we get $\binom{13}{k-2}$ possible sets S. Summing over all allowed values of k, we then have $\binom{13}{0} + \binom{13}{1} + \binom{13}{2} + \cdots + \binom{13}{13-n}$ sets with n as an anchor. If we sum over all $n = 1, 2, \ldots, 13$ (since there are no possible values of k when $n > 13$), we get a total of

$$13\binom{13}{0} + 12\binom{13}{1} + 11\binom{13}{2} + \cdots + \binom{13}{12}.$$

   If we call this quantity A, then, by symmetry, $2A$ equals

$$\begin{array}{rcccccccccccl} 13\binom{13}{0} & + & 12\binom{13}{1} & + & 11\binom{13}{2} & + & \cdots & + & \binom{13}{12} & & & & \\ & + & \binom{13}{1} & + & 2\binom{13}{2} & + & \cdots & + & 12\binom{13}{12} & + & 13\binom{13}{13} & & \\ = 13\left[\binom{13}{0}\right. & + & \binom{13}{1} & + & \binom{13}{2} & + & \cdots & + & \binom{13}{12} & + & \left.\binom{13}{13}\right] & = & 13 \cdot 2^{13}. \end{array}$$

   So $A = 13 \cdot 2^{12}$ is the total number of anchors over all possible sets S. Finally, to find the average number of anchors, we divide by the number of sets, which is 2^{15}; thus, the answer is $13 \cdot 2^{12}/2^{15} = 13/8$.

9. At a certain college, there are 10 clubs and some number of students. For any two different students, there is some club such that exactly one of the two belongs to that club. For any three different students, there is some club such that either exactly one or all three belong to that club. What is the largest possible number of students?

   Solution: $\boxed{513}$

   Let C be the set of clubs; each student then corresponds to a subset of C (the clubs to which that student belongs). The two-student condition implies that these subsets must be all distinct. Now (assuming there is more than one student) some student belongs to a nonempty set S of clubs. For every subset $T \subseteq C$, let $f(T)$ be the subset of C consisting of those clubs that are in exactly one of S and T (so that $f(T) = (S \cup T) - (S \cap T)$). It is straightforward to check that $f(f(T)) = T$ and $f(T) \neq T$, so that the collection of all 2^{10} subsets of C is partitioned into pairs $\{T, f(T)\}$. Moreover, as long as S is distinct from T and $f(T)$, every club is in either none or exactly two of the sets S, T, and $f(T)$, so we cannot have a student corresponding to T and another corresponding to $f(T)$. This puts an upper bound of 513 possible students (one for S, one for $\emptyset = f(S)$, and one for each of the 511 other pairs). On the other hand, if we take some club c, we can have one student belonging to no clubs and 512 other students all belonging to c and to the 512 possible subsets of the other 9 clubs, respectively. It is readily checked that this arrangement meets the conditions — for the three-student condition, either all three students are in c, or one is the student who belongs to no clubs and we reduce to the two-student condition — so 513 is achievable.

10. A calculator has a display, which shows a nonnegative integer N, and a button, which replaces N by a random integer chosen uniformly from the set $\{0, 1, \ldots, N-1\}$, provided that $N > 0$. Initially, the display holds the number $N = 2003$. If the button is pressed repeatedly until $N = 0$, what is the probability that the numbers $1, 10, 100,$ and 1000 will each show up on the display at some point?

    Solution: $\boxed{1/2224222}$

    First, we claim that if the display starts at some N, the probability that any given number $M < N$ will appear at some point is $1/(M+1)$. We can show this by induction on N. If $N = M+1$ (the base case), M can only be reached if it appears after the first step, and this occurs with probability $1/N = 1/(M+1)$. If $N > M+1$ and the claim holds for $N-1$, then there are two possibilities starting from N. If the first step leads to $N-1$ (this occurs with probability $1/N$), the probability of seeing M subsequently is $1/(M+1)$ by the induction hypothesis. If the first step leads to something less than $N-1$ (probability $(N-1)/N$), then it leads to any of the integers $\{0, 1, \ldots, N-2\}$ with equal probability. But this is exactly what the first step would have been if we had started from $N-1$; hence, the probability of seeing M is again $1/(M+1)$ by induction. Thus, the overall probability of seeing M is $\frac{1}{N} \cdot \frac{1}{M+1} + \frac{N-1}{N} \cdot \frac{1}{M+1} = 1/(M+1)$, proving the induction step and the claim.

    Now let $P(N, M)$ $(M < N)$ be the probability of eventually seeing the number M if we start at N; note that this is the same as the conditional probability of seeing M given that we see N. Hence, the desired probability is

$$P(2003, 1000) \cdot P(1000, 100) \cdot P(100, 10) \cdot P(10, 1) = \frac{1}{1001} \cdot \frac{1}{101} \cdot \frac{1}{11} \cdot \frac{1}{2} = \frac{1}{2224222}.$$