Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Algebra Subject Test

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|$$

2. How many real numbers x are solutions to the following equation?

$$2003^x + 2004^x = 2005^x$$

3. Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of

$$\frac{xy+yz+zx}{x^2+y^2+z^2}.$$

4. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

5. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

6. Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

7. Let x be a positive real number. Find the maximum possible value of

$$\frac{x^2 + 2 - \sqrt{x^4+4}}{x}.$$

8. Compute

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

9. The number 27,000,001 has exactly four prime factors. Find their sum.

10. Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.