Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Algebra Subject Test — Solutions

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|$$

Solution: $\boxed{2}$

If $x < 1$, the equation becomes $(1-x) = (2-x) + (3-x)$ which simplifies to $x = 4$, contradicting the assumption $x < 1$. If $1 \le x \le 2$, we get $(x-1) = (2-x) + (3-x)$, which gives $x = 2$. If $2 \le x \le 3$, we get $(x-1) = (x-2) + (3-x)$, which again gives $x = 2$. If $x \ge 3$, we get $(x-1) = (x-2) + (x-3)$, or $x = 4$. So 2 and 4 are the only solutions, and the answer is 2.

2. How many real numbers x are solutions to the following equation?

$$2003^x + 2004^x = 2005^x$$

Solution: $\boxed{1}$

Rewrite the equation as $(2003/2005)^x + (2004/2005)^x = 1$. The left side is strictly decreasing in x, so there cannot be more than one solution. On the other hand, the left side equals $2 > 1$ when $x = 0$ and goes to 0 when x is very large, so it must equal 1 somewhere in between. Therefore there is one solution.

3. Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

Solution: $\boxed{-1/2}$

Note that $0 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. Rearranging, we get that $xy + yz + zx = -\frac{1}{2}(x^2 + y^2 + z^2)$, so that in fact the quantity is always equal to $-1/2$.

4. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

Solution: $\boxed{4}$

Since $\lfloor x \rfloor > x - 1$ for all x, we have that

$$\begin{aligned}
\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor &> \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - 3 \\
&= \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) - 3.
\end{aligned}$$

But by the AM-GM inequality, each of the first three terms in the last line is at least 2. Therefore, the lefthand side is greater than $2 + 2 + 2 - 3 = 3$. Since it is an integer, the smallest value it can be is 4. This is in fact attainable by letting $(a, b, c) = (6, 8, 9)$.

5. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

Solution: $\boxed{28}$

First note that all the integers must be at least 2, because the greatest common divisor of any two positive integers is at least 1. Let n be the largest integer in the circle. The greatest common divisor of its two neighbors is $n-1$. Therefore, each of the two neighbors is at least $n-1$ but at most n, so since $n-1 \nmid n$ for $n-1 \geq 2$, they must both be equal to $n-1$. Let m be one of the numbers on the other side of $n-1$ from n. Then $\gcd(n, m) = n-2$. Since $n-2 \geq 0$, $n-2 \mid n$ only for $n = 3$ or 4. If $n = 3$, each number must be 2 or 3, and it is easy to check that there is no solution. If $n = 4$, then it is again not hard to find that there is a unique solution up to rotation, namely 4322343223. The only possible sum is therefore 28.

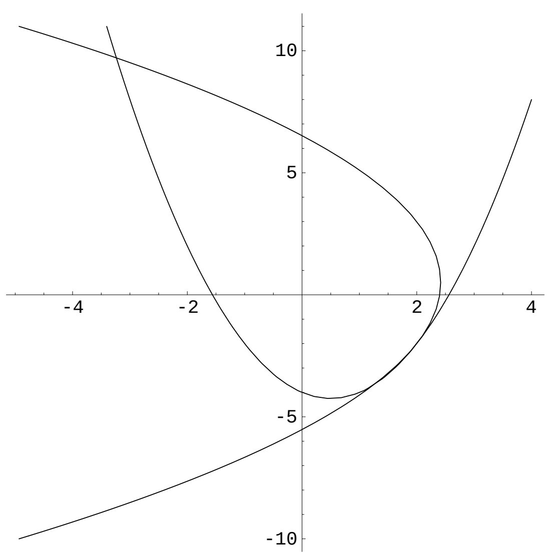
6. Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

Solution: $\boxed{0}$

Substituting $y = x^2 - x - 4$ into the second equation yields

$$\begin{aligned} 0 &= (x^2-x-4)^2 - (x^2-x-4) + 15x - 36 \\ &= x^4 - 2x^3 - 7x^2 + 8x + 16 - x^2 + x + 4 + 15x - 36 \\ &= x^4 - 2x^3 - 8x^2 + 24x - 16 \\ &= (x-2)(x^3 - 8x + 8) = (x-2)^2(x^2 + 2x - 4). \end{aligned}$$

This quartic has three distinct real roots at $x = 2, -1 \pm \sqrt{5}$. Each of these yields a distinct point of intersection, so the answer is their sum, 0.



7. Let x be a positive real number. Find the maximum possible value of

$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}.$$

Solution: $\boxed{2\sqrt{2}-2}$

Rationalizing the numerator, we get

$$\begin{aligned}
\frac{x^2+2-\sqrt{x^4+4}}{x}\cdot\frac{x^2+2+\sqrt{x^4+4}}{x^2+2+\sqrt{x^4+4}} &= \frac{(x^2+2)^2-(x^4+4)}{x(x^2+2+\sqrt{x^4+4})} \\
&= \frac{4x^2}{x(x^2+2+\sqrt{x^4+4})} \\
&= \frac{4}{\frac{1}{x}(x^2+2+\sqrt{x^4+4})} \\
&= \frac{4}{x+\frac{2}{x}+\sqrt{x^2+\frac{4}{x^2}}}.
\end{aligned}$$

Since we wish to maximize this quantity, we wish to minimize the denominator. By AM-GM, $x+\frac{2}{x}\geq 2\sqrt{2}$ and $x^2+\frac{4}{x^2}\geq 4$, so that the denominator is at least $2\sqrt{2}+2$. Therefore,

$$\frac{x^2+2-\sqrt{x^4+4}}{x}\leq\frac{4}{2\sqrt{2}+2}=2\sqrt{2}-2,$$

with equality when $x=\sqrt{2}$.

8. Compute

$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}.$$

Solution: $\boxed{1/2}$

Note that

$$n^4+n^2+1=(n^4+2n^2+1)-n^2=(n^2+1)^2-n^2=(n^2+n+1)(n^2-n+1).$$

Decomposing into partial fractions, we find that

$$\frac{n}{n^4+n^2+1}=\frac{1}{2}\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right).$$

Now, note that if $f(n)=\frac{1}{n^2-n+1}$, then $f(n+1)=\frac{1}{(n+1)^2-(n+1)+1}=\frac{1}{n^2+n+1}$. It follows that

$$\sum_{n=0}^{\infty}\frac{n}{n^4+n^2+1}=\frac{1}{2}\Big((f(0)-f(1))+(f(1)-f(2))+(f(2)-f(3))+\cdots\Big).$$

Since $f(n)$ tends towards 0 as n gets large, this sum telescopes to $f(0)/2=1/2$.

9. The number 27,000,001 has exactly four prime factors. Find their sum.

Solution: $\boxed{652}$

First, we factor

$$\begin{aligned}27x^6+1 &= (3x^2)^3+1\\ &= (3x^2+1)(9x^4-3x^2+1)\\ &= (3x^2+1)((9x^4+6x^2+1)-9x^2)\\ &= (3x^2+1)((3x^2+1)^2-(3x)^2)\\ &= (3x^2+1)(3x^2+3x+1)(3x^2-3x+1).\end{aligned}$$

Letting $x=10$, we get that $27000001 = 301\cdot 331\cdot 271$. A quick check shows that $301 = 7\cdot 43$, so that $27000001 = 7\cdot 43\cdot 271\cdot 331$. Each factor here is prime, and their sum is 652.

10. Find the sum of the absolute values of the roots of $x^4-4x^3-4x^2+16x-8=0$.

    Solution: $\boxed{2+2\sqrt{2}+2\sqrt{3}}$

$$\begin{aligned}x^4-4x^3-4x^2+16x-8 &= (x^4-4x^3+4x^2)-(8x^2-16x+8)\\ &= x^2(x-2)^2-8(x-1)^2\\ &= (x^2-2x)^2-(2\sqrt{2}x-2\sqrt{2})^2\\ &= (x^2-(2+2\sqrt{2})x+2\sqrt{2})(x^2-(2-2\sqrt{2})x-2\sqrt{2}).\end{aligned}$$

But noting that $(1+\sqrt{2})^2 = 3+2\sqrt{2}$ and completing the square,

$$\begin{aligned}x^2-(2+2\sqrt{2})x+2\sqrt{2} &= x^2-(2+2\sqrt{2})x+3+2\sqrt{2}-3\\ &= (x-(1+\sqrt{2}))^2-(\sqrt{3})^2\\ &= (x-1-\sqrt{2}+\sqrt{3})(x-1-\sqrt{2}-\sqrt{3}).\end{aligned}$$

Likewise,

$$x^2-(2-2\sqrt{2})x-2\sqrt{2} = (x-1+\sqrt{2}+\sqrt{3})(x-1+\sqrt{2}-\sqrt{3}),$$

so the roots of the quartic are $1\pm\sqrt{2}\pm\sqrt{3}$. Only one of these is negative, namely $1-\sqrt{2}-\sqrt{3}$, so the sum of the absolute values of the roots is

$$(1+\sqrt{2}+\sqrt{3})+(1+\sqrt{2}-\sqrt{3})+(1-\sqrt{2}+\sqrt{3})-(1-\sqrt{2}-\sqrt{3}) = 2+2\sqrt{2}+2\sqrt{3}.$$