Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Calculus Subject Test

1. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

2. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u}\, du$ and $y(t) = \int_t^\infty \frac{\sin u}{u}\, du$ for $1 \leq t \leq 2$. What is the length of the curve?

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x)f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

5. Calculate
$$\lim_{x \to 0^+} \left(x^{x^x} - x^x \right).$$

6. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

7. Two ants, one starting at $(-1, 1)$, the other at $(1, 1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

8. If f is a continuous real function such that $f(x-1) + f(x+1) \geq x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x)dx$?

9. Compute
$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}.$$

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x) = f(1-x)$ for all x and $f(0) = 1$. Find $f(1)$.