Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Calculus Subject Test — Solutions

1. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

   Solution: $\boxed{1}$

   Since $f'(x) = 3x^2 + a$, we must have $3a^2 + a = 3b^2 + a$. Then $a^2 = b^2$, and since $a \neq b$, $a = -b$. Thus $f(1) = 1 + a + b = 1$.

2. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u}\, du$ and $y(t) = \int_t^\infty \frac{\sin u}{u}\, du$ for $1 \leq t \leq 2$. What is the length of the curve?

   Solution: $\boxed{\ln 2}$

   By the Second Fundamental Theorem of Calculus, $\frac{dx}{dt} = -\frac{\cos t}{t}$ and $\frac{dy}{dt} = -\frac{\sin t}{t}$. Therefore, the length of the curve is

$$\int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\, dt = \int_1^2 \frac{1}{t}\, dt = [\ln t]_1^2 = \ln 2.$$

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

   Solution: $\boxed{486/5}$

$$0 = \int_0^1 f(x)f'(x)dx = \int_{f(0)}^{f(1)} u\, du = \frac{1}{2}(f(1)^2 - f(0)^2), \text{ and}$$

$$18 = \int_0^1 f(x)^2 f'(x)dx = \int_{f(0)}^{f(1)} u^2 du = \frac{1}{3}(f(1)^3 - f(0)^3).$$

   The first equation implies $f(0) = \pm f(1)$. The second equation shows that $f(0) \neq f(1)$, and in fact $54 = f(1)^3 - f(0)^3 = 2f(1)^3$, so $f(1) = 3$ and $f(0) = -3$. Then

$$\int_0^1 f(x)^4 f'(x)dx = \int_{f(0)}^{f(1)} u^4 du = \frac{1}{5}(f(1)^5 - f(0)^5) = \frac{1}{5}(243 + 243) = \frac{486}{5}.$$

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x)f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

   Solution: $\boxed{\pm\sqrt{3}}$

   Let $f'(0) = a$. Then the equation gives $f''(0) = a^2$. Differentiating the given equation gives

$$2f'(x)f''(x) = f(x)f'''(x) + f'(x)f''(x),$$

   or $f'(x)f''(x) = f(x)f'''(x)$. Differentiating once more gives

$$f'(x)f'''(x) + f''(x)^2 = f(x)f^{(4)}(x) + f'(x)f'''(x)$$

or $f''(x)^2 = f(x)f^{(4)}(x)$, giving $9 = f^{(4)}(0) = a^4$. Thus $a = \pm\sqrt{3}$. These are indeed both attainable by $f(x) = e^{\pm x\sqrt{3}}$.

Alternative Solution: Rewrite the given equation as $\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$. Integrating both sides gives $\ln f'(x) = \ln f(x) + C_1$, and exponentiating gives $f'(x) = Cf(x)$. This has solution $f(x) = Ae^{Cx}$ for constants A and C. Since $f(0) = 1$, $A = 1$, and differentiating we find that $C^4 = f^{(4)}(0) = 9$, yielding $f'(0) = C = \pm\sqrt{3}$.

5. Calculate

$$\lim_{x\to 0^+} \left(x^{x^x} - x^x\right).$$

Solution: $\boxed{-1}$

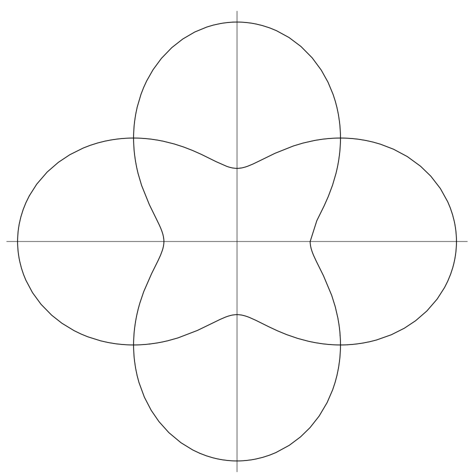
We first calculate $\lim_{x\to 0^+} x^x$: it is just $\exp(\lim_{x\to 0^+} x\ln x)$. But

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

by L'Hôpital's Rule. Therefore $\lim_{x\to 0^+} x^x = 1$. Then $\lim_{x\to 0^+} x^{x^x} = 0^1 = 0$, so $\lim_{x\to 0^+} \left(x^{x^x} - x^x\right) = -1$.

6. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

Solution: $\boxed{9\pi/2 - 8}$



The original graph is closer to the origin than its reflection for $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})$, and the region is symmetric about the origin. Therefore the area we wish to find is the polar integral

$$\begin{aligned}
4\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2}(2+\cos 2\theta)^2 d\theta &= 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 + 4\cos 2\theta + \cos^2 2\theta) d\theta \\
&= 2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(4 + 4\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right) d\theta \\
&= \left[9\theta + 4\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
&= \left(\frac{27\pi}{4} - 4\right) - \left(\frac{9\pi}{4} + 4\right) = \frac{9\pi}{2} - 8.
\end{aligned}$$

7. Two ants, one starting at $(-1,1)$, the other at $(1,1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

Solution: $\boxed{3\sqrt{3}-3}$

When the left ant first hits the line $y = \frac{1}{2}$, the right ant hits the line $y = \frac{3}{2}$. The left ant is then at $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$, and the right ant is at $(\frac{\sqrt{6}}{2}, \frac{3}{2})$. Let the left ant have velocity with components v_x and v_y, the right ant velocity with components w_x and w_y. Since $\frac{dy}{dx} = 2x$, $\frac{v_y}{v_x} = -\sqrt{2}$ and $\frac{w_y}{w_x} = \sqrt{6}$. Since the midpoint of the ants moves at speed 1 along the line $y = 1$, $\frac{1}{2}(v_x + w_x) = 1$ and $\frac{1}{2}(v_y + w_y) = 0$. Therefore, $\sqrt{2}v_x = -v_y = w_y = \sqrt{6}w_x = \sqrt{6}(2 - v_x)$. Solving for v_x gives $\frac{2\sqrt{6}}{\sqrt{6}+\sqrt{2}} = 3 - \sqrt{3}$. Then the speed of the left ant is

$$\sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + (-\sqrt{2}v_x)^2} = \sqrt{3}|v_x| = 3\sqrt{3} - 3.$$

8. If f is a continuous real function such that $f(x-1) + f(x+1) \geq x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x)dx$?

Solution: $\boxed{2010012}$

Let $g(x) = f(x) - x$. Then

$$g(x-1) + x - 1 + g(x+1) + x + 1 \geq x + g(x) + x,$$

or $g(x-1) + g(x+1) \geq g(x)$. But now,

$$g(x+3) \geq g(x+2) - g(x+1) \geq -g(x).$$

Therefore

$$\begin{aligned} \int_a^{a+6} g(x)dx &= \int_a^{a+3} g(x)dx + \int_{a+3}^{a+6} g(x)dx \\ &= \int_a^{a+3} (g(x) + g(x+3))dx \geq 0. \end{aligned}$$

It follows that

$$\int_1^{2005} g(x) = \sum_{n=0}^{333} \int_{6n+1}^{6n+7} g(x)dx \geq 0,$$

so that

$$\int_1^{2005} f(x)dx = \int_1^{2005} (g(x) + x)dx \geq \int_1^{2005} x\,dx = \left[\frac{x^2}{2}\right]_1^{2005} = \frac{2005^2 - 1}{2} = 2010012.$$

Equality holds for $f(x) = x$.

9. Compute

$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}.$$

Solution: $\boxed{e + 1/e + 2\cos 1}$

This is the power series

$$4 + \frac{4x^4}{4!} + \frac{4x^8}{8!} + \cdots$$

evaluated at $x = 1$. But this power series can be written as the sum

$$\begin{aligned}
& \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\right) \\
+ \quad & \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right) \\
+ \quad & 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \\
= \quad & e^x + e^{-x} + 2\cos x.
\end{aligned}$$

It follows that the quantity is $e + 1/e + 2\cos 1$.

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x) = f(1-x)$ for all x and $f(0) = 1$. Find $f(1)$.

Solution: $\boxed{\sec 1 + \tan 1}$

Differentiating the given equation gives $f''(x) = -f(x)$. This has solutions of the form $A\cos(x) + B\sin(x)$. Since $f(0) = 1$, $A = 1$. Then $f'(x) = B\cos(x) - \sin(x)$ and

$$\begin{aligned}
f(1-x) &= \cos(1-x) + B\sin(1-x) \\
&= \cos 1 \cos x + \sin 1 \sin x + B \sin 1 \cos x - B \cos 1 \sin x \\
&= (\cos 1 + B \sin 1)\cos x + (\sin 1 - B\cos 1)\sin x.
\end{aligned}$$

Therefore, $B = \cos 1 + B\sin 1$ and $-1 = \sin 1 - B\cos 1$, both of which yield as solutions

$$B = \frac{\cos 1}{1 - \sin 1} = \frac{1 + \sin 1}{\cos 1} = \sec 1 + \tan 1.$$