Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Combinatorics Subject Test

1. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

2. How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

3. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

4. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

5. Doug and Ryan are competing in the 2005 Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $1/3$. When Ryan swings, the probability that he will hit a home run is $1/2$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

6. Three fair six-sided dice, each numbered 1 through 6, are rolled. What is the probability that the three numbers that come up can form the sides of a triangle?

7. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

8. Every second, Andrea writes down a random digit uniformly chosen from the set $\{1, 2, 3, 4\}$. She stops when the last two numbers she has written sum to a prime number. What is the probability that the last number she writes down is 1?

9. Eight coins are arranged in a circle heads up. A move consists of flipping over two adjacent coins. How many different sequences of six moves leave the coins alternating heads up and tails up?

10. You start out with a big pile of 3^{2004} cards, with the numbers $1, 2, 3, \ldots, 3^{2004}$ written on them. You arrange the cards into groups of three any way you like; from each group, you keep the card with the largest number and discard the other two. You now again arrange these 3^{2003} remaining cards into groups of three any way you like, and in each group, keep the card with the smallest number and discard the other two. You now have 3^{2002} cards, and you again arrange these into groups of three and keep the largest number in each group. You proceed in this manner, alternating between keeping the largest number and keeping the smallest number in each group, until you have just one card left.

    How many different values are possible for the number on this final card?