Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Combinatorics Subject Test — Solutions

1. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

   Solution: $\boxed{22}$

   Suppose that either nine or ten of the questions have the same answer. Then no matter which five questions we pick to have this answer, we will be right at least four times. Conversely, suppose that there are at least two questions with each answer; we will show that we can get a score less than four. By symmetry, assume there are at least five questions whose answer is true. Then if we label five of these false, not only will we get these five wrong, but we will also have answered all the false questions with true, for a total of at least seven incorrect. There are 2 ways for all the questions to have the same answer, and $2 \cdot 10 = 20$ ways for one question to have a different answer from the others, for a total of 22 ways.

2. How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

   Solution: $\boxed{1365}$

   If a is the smallest element of such a set, then $13 - a$ is the largest element, and for the remaining elements we may choose any (or none) of the $12 - 2a$ elements $a+1, a+2, \ldots, (13-a)-1$. Thus there are 2^{12-2a} such sets whose smallest element is a. Also, $13 - a \geq a$ clearly implies $a < 7$. Summing over all $a = 1, 2, \ldots, 6$, we get a total of

   $$2^{10} + 2^8 + 2^6 + \cdots + 2^0 = 4^5 + 4^4 + \cdots + 4^0 = (4^6 - 1)/(4 - 1) = 4095/3 = 1365$$

   possible sets.

3. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

   Solution: $\boxed{1/2}$

   Note that if we imagine that the series always continues to seven games even after one team has won four, this will never change the winner of the series. Notice also that the probability that the Red Sox will win Game n is precisely the probability that the Yankees will win Game $8 - n$. Therefore, the probability that the Yankees win at least four games is the same as the probability that the Red Sox win at least four games, namely $1/2$.

4. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

Solution: $\boxed{216}$

There are $4! = 24$ ways to place the four purple balls into the grid. Choose any purple ball, and place two green balls, one in its row and the other in its column. There are four boxes that do not yet lie in the same row or column as a green ball, and at least one of these contains a purple ball (otherwise the two rows containing green balls would contain the original purple ball as well as the two in the columns not containing green balls). It is then easy to see that there is a unique way to place the remaining green balls. Therefore, there are a total of $24 \cdot 9 = 216$ ways.

5. Doug and Ryan are competing in the 2005 Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $1/3$. When Ryan swings, the probability that he will hit a home run is $1/2$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

   Solution: $\boxed{1/5}$

   Denote this probability by p. Doug hits more home runs if he hits a home run on his first try when Ryan does not, or if they both hit home runs on their first try and Doug hits more home runs thereafter. The probability of the first case occurring is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$, and the probability of the second case occurring is $\frac{1}{3} \cdot \frac{1}{2} \cdot p = \frac{p}{6}$. Therefore $p = \frac{1}{6} + \frac{p}{6}$, which we solve to find $p = \frac{1}{5}$.

6. Three fair six-sided dice, each numbered 1 through 6, are rolled. What is the probability that the three numbers that come up can form the sides of a triangle?

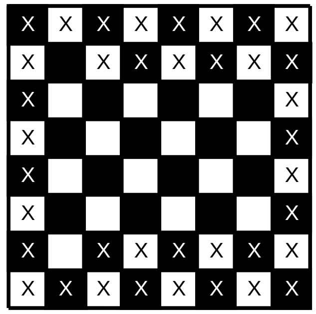
   Solution: $\boxed{37/72}$

   Denote this probability by p, and let the three numbers that come up be x, y, and z. We will calculate $1-p$ instead: $1-p$ is the probability that $x \geq y+z$, $y \geq z+x$, or $z \geq x+y$. Since these three events are mutually exclusive, $1 - p$ is just 3 times the probability that $x \geq y + z$. This happens with probability $(0+1+3+6+10+15)/216 = 35/216$, so the answer is $1 - 3 \cdot (35/216) = 1 - 35/72 = 37/72$.

7. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

   Solution: $\boxed{38}$

   If the chessboard is colored black and white as usual, then any diagonal is a solid color, so we may consider bishops on black and white squares separately. In one direction, the lengths of the black diagonals are 2, 4, 6, 8, 6, 4, and 2. Each of these can have at most three bishops, except the first and last which can have at most two, giving a total of at most $2+3+3+3+3+3+2 = 19$ bishops on black squares. Likewise there can be at most 19 bishops on white squares for a total of at most 38 bishops. This is indeed attainable as in the diagram below.



8. Every second, Andrea writes down a random digit uniformly chosen from the set $\{1, 2, 3, 4\}$. She stops when the last two numbers she has written sum to a prime number. What is the probability that the last number she writes down is 1?

   Solution: $\boxed{15/44}$

   Let p_n be the probability that the last number she writes down is 1 when the first number she writes down is n. Suppose she starts by writing 2 or 4. Then she can continue writing either 2 or 4, but the first time she writes 1 or 3, she stops. Therefore $p_2 = p_4 = \frac{1}{2}$. Suppose she starts by writing 1. Then she stops if she writes 1, 2, or 4, but continues if she writes 3. Therefore $p_1 = \frac{1}{4}(1 + p_3)$. If she starts by writing 3, then she stops if she writes 2 or 4 and otherwise continues. Therefore $p_3 = \frac{1}{4}(p_1 + p_3) = \frac{1}{16}(1 + 5p_3)$. Solving gives $p_3 = \frac{1}{11}$ and $p_1 = \frac{3}{11}$. The probability we want to find is therefore $\frac{1}{4}(p_1 + p_2 + p_3 + p_4) = \frac{15}{44}$.

9. Eight coins are arranged in a circle heads up. A move consists of flipping over two adjacent coins. How many different sequences of six moves leave the coins alternating heads up and tails up?

   Solution: $\boxed{7680}$

   Imagine we flip over two adjacent coins by pushing a button halfway between them. Then the outcome depends only on the parities of the number of times that each button is pushed. To flip any coin, we must push the two buttons adjacent to that coin a total of an odd number of times. To flip every other coin, the parities must then progress around the circle as even, even, odd, odd, even, even, odd, odd. There are 4 ways to assign these parities. If we assume each button is pressed either once or not at all, this accounts for only four presses, so some button is also pressed twice more. Suppose this button was already pushed once. There are 4 of these, and the number of possible sequences of presses is then $6!/3! = 120$. Suppose it has not already been pressed. There are 4 of these as well, and the number of possible sequences is $6!/2! = 360$. The total number of sequences is then $4(4 \cdot 120 + 4 \cdot 360) = 7680$.

10. You start out with a big pile of 3^{2004} cards, with the numbers $1, 2, 3, \ldots, 3^{2004}$ written on them. You arrange the cards into groups of three any way you like; from each group, you keep the card with the largest number and discard the other two. You now again arrange these 3^{2003} remaining cards into groups of three any way you like, and in each group, keep the card with the smallest number and discard the other two. You now have 3^{2002} cards, and you again arrange these into groups of three and keep the largest number in each group. You proceed in this manner, alternating between keeping the largest number and keeping the smallest number in each group, until you have just one card left.

How many different values are possible for the number on this final card?

Solution: $\boxed{3^{2004} - 2 \cdot 3^{1002} + 2}$

We claim that if you have cards numbered $1, 2, \ldots, 3^{2n}$ and perform $2n$ successive grouping operations, then c is a possible value for your last remaining card if and only if

$$3^n \le c \le 3^{2n} - 3^n + 1.$$

This gives $3^{2n} - 2 \cdot 3^n + 2$ possible values of c, for a final answer of $3^{2004} - 2 \cdot 3^{1002} + 2$.

Indeed, notice that the last remaining card c must have been the largest of some set of three at the $(2n-1)$th step; each of these was in turn the largest of some set of three (and so c was the largest of some set of 9 cards) remaining at the $(2n-3)$th step; each of these was in turn the largest of some set of three (and so c was the largest of some set of 27) remaining at the $(2n-5)$th step; continuing in this manner, we get that c was the largest of some 3^n cards at the first step, so $c \ge 3^n$. A similar analysis of all of the steps in which we save the smallest card gives that c is the smallest of some set of 3^n initial cards, so $c \le 3^{2n} - 3^n + 1$.

To see that any c in this interval is indeed possible, we will carry out the groupings inductively so that, after $2i$ steps, the following condition is satisfied: if the numbers remaining are $a_1 < a_2 < \cdots < a_{3^{2(n-i)}}$, then c is one of these, and there are at least $3^{n-i} - 1$ numbers smaller than c and at least $3^{n-i} - 1$ numbers larger than c. This is certainly true when $i = 0$, so it suffices to show that if it holds for some $i < n$, we can perform the grouping so that the condition will still hold for $i + 1$. Well, we first group the smallest numbers as $\{a_1, a_2, a_3\}, \{a_4, a_5, a_6\}, \ldots, \{a_{3^{n-i}-5}, a_{3^{n-i}-4}, a_{3^{n-i}-3}\}$. We then group the remaining numbers in such a way that c and the largest $3^{n-i} - 1$ numbers are each the largest in its respective group; it is easy to see that we can do this. After retaining the largest number in each group, we will then have at least $3^{n-i-1} - 1$ numbers smaller than c remaining and at least $3^{n-i} - 1$ numbers larger than c remaining. And for the next grouping, we similarly group the largest $3^{n-i} - 3$ numbers into $3^{n-i-1} - 1$ groups, and arrange the remaining numbers so that the smallest $3^{n-i-1} - 1$ numbers and c are all the smallest in their groups. After this round of discarding, then c will be retained, and we will still have at least $3^{n-i-1} - 1$ numbers larger than c and 3^{n-i-1} numbers smaller than c. This proves the induction step, and now the solution is complete.