Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Geometry Subject Test

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

3. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD. What is the area of the triangle formed by the centroids of triangles ABE, BCE, and ADE?

4. Let XYZ be a triangle with $\angle X = 60^\circ$ and $\angle Y = 45^\circ$. A circle with center P passes through points A and B on side XY, C and D on side YZ, and E and F on side ZX. Suppose $AB = CD = EF$. Find $\angle XPY$ in degrees.

5. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

6. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

7. Let $ABCD$ be a tetrahedron such that edges AB, AC, and AD are mutually perpendicular. Let the areas of triangles ABC, ACD, and ADB be denoted by x, y, and z, respectively. In terms of x, y, and z, find the area of triangle BCD.

8. Let T be a triangle with side lengths 26, 51, and 73. Let S be the set of points inside T which do not lie within a distance of 5 of any side of T. Find the area of S.

9. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = 1/5$. Let B be the point on ω such that DB is perpendicular to AC, and let E be the midpoint of DB. The line tangent to ω at B intersects line CE at the point X. Compute AX.

10. Let AB be the diameter of a semicircle Γ. Two circles, ω_1 and ω_2, externally tangent to each other and internally tangent to Γ, are tangent to the line AB at P and Q, respectively, and to semicircular arc AB at C and D, respectively, with $AP < AQ$. Suppose F lies on Γ such that $\angle FQB = \angle CQA$ and that $\angle ABF = 80^\circ$. Find $\angle PDQ$ in degrees.