Harvard-MIT Mathematics Tournament

February 19, 2005

Guts Round

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1. **[5]** Find the largest positive integer n such that $1+2+3+\cdots+n^2$ is divisible by $1+2+3+\cdots+n$.

2. **[5]** Let x, y, and z be positive real numbers such that $(x\cdot y)+z=(x+z)\cdot(y+z)$. What is the maximum possible value of xyz?

3. **[5]** Find the sum
$$\frac{2^1}{4^1-1}+\frac{2^2}{4^2-1}+\frac{2^4}{4^4-1}+\frac{2^8}{4^8-1}+\cdots.$$

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4. **[6]** What is the probability that in a randomly chosen arrangement of the numbers and letters in "HMMT2005," one can read either "HMMT" or "2005" from left to right? (For example, in "5HM0M20T," one can read "HMMT.")

5. **[6]** For how many integers n between 1 and 2005, inclusive, is $2\cdot 6\cdot 10\cdots(4n-2)$ divisible by $n!$?

6. **[6]** Let $m\circ n=(m+n)/(mn+4)$. Compute $((\cdots((2005\circ 2004)\circ 2003)\circ\cdots\circ 1)\circ 0)$.

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7. **[6]** Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

8. **[6]** Let $ABCD$ be a convex quadrilateral inscribed in a circle with shortest side AB. The ratio $[BCD]/[ABD]$ is an integer (where $[XYZ]$ denotes the area of triangle XYZ.) If the lengths of AB, BC, CD, and DA are distinct integers no greater than 10, find the largest possible value of AB.

9. **[6]** Farmer Bill's 1000 animals — ducks, cows, and rabbits — are standing in a circle. In order to feel safe, every duck must either be standing next to at least one cow or between two rabbits. If there are 600 ducks, what is the least number of cows there can be for this to be possible?

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10. [**7**] You are given a set of cards labeled from 1 to 100. You wish to make piles of three cards such that in any pile, the number on one of the cards is the product of the numbers on the other two cards. However, no card can be in more than one pile. What is the maximum number of piles you can form at once?

11. [**7**] The Dingoberry Farm is a 10 mile by 10 mile square, broken up into 1 mile by 1 mile patches. Each patch is farmed either by Farmer Keith or by Farmer Ann. Whenever Ann farms a patch, she also farms all the patches due west of it and all the patches due south of it. Ann puts up a scarecrow on each of her patches that is adjacent to exactly two of Keith's patches (and nowhere else). If Ann farms a total of 30 patches, what is the largest number of scarecrows she could put up?

12. [**7**] Two vertices of a cube are given in space. The locus of points that could be a third vertex of the cube is the union of n circles. Find n.

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13. [**7**] Triangle ABC has $AB = 1$, $BC = \sqrt{7}$, and $CA = \sqrt{3}$. Let ℓ_1 be the line through A perpendicular to AB, ℓ_2 the line through B perpendicular to AC, and P the point of intersection of ℓ_1 and ℓ_2. Find PC.

14. [**7**] Three noncollinear points and a line ℓ are given in the plane. Suppose no two of the points lie on a line parallel to ℓ (or ℓ itself). There are exactly n lines perpendicular to ℓ with the following property: the three circles with centers at the given points and tangent to the line all concur at some point. Find all possible values of n.

15. [**7**] Let S be the set of lattice points inside the circle $x^2 + y^2 = 11$. Let M be the greatest area of any triangle with vertices in S. How many triangles with vertices in S have area M?

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16. [**8**] A regular octahedron has a side length of 1. What is the distance between two opposite faces?

17. [**8**] Compute

$$2\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2\cdots}}}}.$$

18. [**8**] If a, b, and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a, b, and c?

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19. [**8**] Regular tetrahedron $ABCD$ is projected onto a plane sending A, B, C, and D to A', B', C', and D' respectively. Suppose $A'B'C'D'$ is a convex quadrilateral with $A'B' = B'C$' and $C'D' = D'A'$, and suppose that the area of $A'B'C'D' = 4$. Given these conditions, the set of possible lengths of AB consists of all real numbers in the interval $[a, b)$. Compute b.

20. [**8**] If n is a positive integer, let $s(n)$ denote the sum of the digits of n. We say that n is *zesty* if there exist positive integers x and y greater than 1 such that $xy = n$ and $s(x)s(y) = s(n)$. How many zesty two-digit numbers are there?

21. [**8**] In triangle ABC with altitude AD, $\angle BAC = 45°$, $DB = 3$, and $CD = 2$. Find the area of triangle ABC.

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22. [**9**] Find

$$\{\ln(1+e)\} + \{\ln(1+e^2)\} + \{\ln(1+e^4)\} + \{\ln(1+e^8)\} + \cdots ,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

23. [**9**] The sides of a regular hexagon are trisected, resulting in 18 points, including vertices. These points, starting with a vertex, are numbered clockwise as $A_1, A_2, \ldots, A_{18}$. The line segment A_kA_{k+4} is drawn for $k = 1, 4, 7, 10, 13, 16$, where indices are taken modulo 18. These segments define a region containing the center of the hexagon. Find the ratio of the area of this region to the area of the large hexagon.

24. [**9**] In the base 10 arithmetic problem $HMMT + GUTS = ROUND$, each distinct letter represents a different digit, and leading zeroes are not allowed. What is the maximum possible value of $ROUND$?

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25. **[9]** An ant starts at one vertex of a tetrahedron. Each minute it walks along a random edge to an adjacent vertex. What is the probability that after one hour the ant winds up at the same vertex it started at?

26. **[9]** In triangle ABC, $AC = 3AB$. Let AD bisect angle A with D lying on BC, and let E be the foot of the perpendicular from C to AD. Find $[ABD]/[CDE]$. (Here, $[XYZ]$ denotes the area of triangle XYZ).

27. **[9]** In a chess-playing club, some of the players take lessons from other players. It is possible (but not necessary) for two players both to take lessons from each other. It so happens that for any three distinct members of the club, A, B, and C, exactly one of the following three statements is true: A takes lessons from B; B takes lessons from C; C takes lessons from A. What is the largest number of players there can be?

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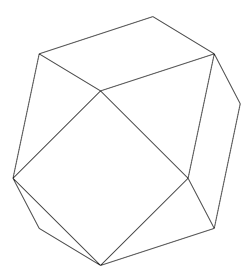
28. **[10]** There are three pairs of real numbers (x_1, y_1), (x_2, y_2), and (x_3, y_3) that satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$. Compute $\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_2}\right)\left(1 - \frac{x_3}{y_3}\right)$.

29. **[10]** Let $n > 0$ be an integer. Each face of a regular tetrahedron is painted in one of n colors (the faces are not necessarily painted different colors.) Suppose there are n^3 possible colorings, where rotations, but not reflections, of the same coloring are considered the same. Find all possible values of n.

30. **[10]** A cuboctahedron is a polyhedron whose faces are squares and equilateral triangles such that two squares and two triangles alternate around each vertex, as shown.



What is the volume of a cuboctahedron of side length 1?

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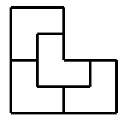
31. **[10]** The L shape made by adjoining three congruent squares can be subdivided into four smaller L shapes.



Each of these can in turn be subdivided, and so forth. If we perform 2005 successive subdivisions, how many of the 4^{2005} L's left at the end will be in the same orientation as the original one?

32. **[10]** Let $a_1 = 3$, and for $n \geq 1$, let $a_{n+1} = (n+1)a_n - n$. Find the smallest $m \geq 2005$ such that $a_{m+1} - 1 \mid a_m^2 - 1$.

33. **[10]** Triangle ABC has incircle ω which touches AB at C_1, BC at A_1, and CA at B_1. Let A_2 be the reflection of A_1 over the midpoint of BC, and define B_2 and C_2 similarly. Let A_3 be the intersection of AA_2 with ω that is closer to A, and define B_3 and C_3 similarly. If $AB = 9$, $BC = 10$, and $CA = 13$, find $[A_3B_3C_3]/[ABC]$. (Here $[XYZ]$ denotes the area of triangle XYZ.)

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34. **[12]** A regular octahedron $ABCDEF$ is given such that AD, BE, and CF are perpendicular. Let G, H, and I lie on edges AB, BC, and CA respectively such that $\frac{AG}{GB} = \frac{BH}{HC} = \frac{CI}{IA} = \rho$. For some choice of $\rho > 1$, GH, HI, and IG are three edges of a regular icosahedron, eight of whose faces are inscribed in the faces of $ABCDEF$. Find ρ.

35. **[12]** Let $p = 2^{24036583} - 1$, the largest prime currently known. For how many positive integers c do each of the quadratics $\pm x^2 \pm px \pm c$ have rational roots?

36. **[12]** One hundred people are in line to see a movie. Each person wants to sit in the front row, which contains one hundred seats, and each has a favorite seat, chosen randomly. They enter the row one at a time from the far right. As they walk, if they reach their favorite seat, they sit, but to avoid stepping over people, if they encounter a person already seated, they sit to that person's right. If the seat furthest to the right is already taken, they sit in a different row. What is the most likely number of people that will get to sit in the first row?

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37. **[15]** Let $a_1, a_2, \ldots, a_{2005}$ be real numbers such that

$$\begin{aligned} a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots + a_{2005} \cdot 2005 &= 0 \\ a_1 \cdot 1^2 + a_2 \cdot 2^2 + a_3 \cdot 3^2 + \cdots + a_{2005} \cdot 2005^2 &= 0 \\ a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_{2005} \cdot 2005^3 &= 0 \\ &\vdots \\ a_1 \cdot 1^{2004} + a_2 \cdot 2^{2004} + a_3 \cdot 3^{2004} + \cdots + a_{2005} \cdot 2005^{2004} &= 0 \end{aligned}$$

and

$$a_1 \cdot 1^{2005} + a_2 \cdot 2^{2005} + a_3 \cdot 3^{2005} + \cdots + a_{2005} \cdot 2005^{2005} = 1.$$

What is the value of a_1?

38. **[15]** In how many ways can the set of ordered pairs of integers be colored red and blue such that for all a and b, the points (a, b), $(-1-b, a+1)$, and $(1-b, a-1)$ are all the same color?

39. **[15]** How many regions of the plane are bounded by the graph of

$$x^6 - x^5 + 3x^4y^2 + 10x^3y^2 + 3x^2y^4 - 5xy^4 + y^6 = 0?$$

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40. **[18]** In a town of n people, a governing council is elected as follows: each person casts one vote for some person in the town, and anyone that receives at least five votes is elected to council. Let $c(n)$ denote the expected number of people elected to council if everyone votes randomly. Find $\lim_{n\to\infty} c(n)/n$.

41. **[18]** There are 42 stepping stones in a pond, arranged along a circle. You are standing on one of the stones. You would like to jump among the stones so that you move counterclockwise by either 1 stone or 7 stones at each jump. Moreover, you would like to do this in such a way that you visit each stone (except for the starting spot) exactly once before returning to your initial stone for the first time. In how many ways can you do this?

42. **[18]** In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid such that every row and column contains two balls of one color and one ball of the other color? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

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43. Write down an integer N between 0 and 20 inclusive. If at least N teams write down N, your score is N; otherwise it is 0.

44. Write down a set S of positive integers, all greater than 1, whose product is P, such that for each $x \in S$, x is a proper divisor of $(P/x)+1$. Your score is $2n$, where $n = |S|$.

45. A *binary word* is a finite sequence of 0's and 1's. A *square subword* is a subsequence consisting of two identical chunks next to each other. For example, the word 100101011 contains the square subwords 00, 0101 (twice), 1010, and 11.

    Find a long binary word containing a small number of square subwords. Specifically, write down a binary word of any length $n \leq 50$. Your score will be $\max\{0, n - s\}$, where s is the number of occurrences of square subwords. (That is, each different square subword will be counted according to the number of times it appears.)