Harvard-MIT Mathematics Tournament

February 19, 2005

Guts Round — Solutions

1. Find the largest positive integer n such that $1+2+3+\cdots+n^2$ is divisible by $1+2+3+\cdots+n$.

   Solution: $\boxed{1}$

   The statement is

   $$\frac{n(n+1)}{2} \mid \frac{n^2(n^2+1)}{2} \Leftrightarrow n+1 \mid n(n^2+1) = n^3+n.$$

   But $n+1$ also divides $(n+1)(n^2-n+2) = n^3+n+2$, so $n+1$ must divide 2. Hence, n cannot be greater than 1. And $n=1$ clearly works, so that is the answer.

2. Let x, y, and z be positive real numbers such that $(x \cdot y) + z = (x+z) \cdot (y+z)$. What is the maximum possible value of xyz?

   Solution: $\boxed{1/27}$

   The condition is equivalent to $z^2 + (x+y-1)z = 0$. Since z is positive, $z = 1-x-y$, so $x+y+z = 1$. By the AM-GM inequality,

   $$xyz \leq \left(\frac{x+y+z}{3}\right)^3 = \frac{1}{27},$$

   with equality when $x=y=z=\frac{1}{3}$.

3. Find the sum

   $$\frac{2^1}{4^1-1} + \frac{2^2}{4^2-1} + \frac{2^4}{4^4-1} + \frac{2^8}{4^8-1} + \cdots.$$

   Solution: $\boxed{1}$

   Notice that

   $$\frac{2^{2^k}}{4^{2^k}-1} = \frac{2^{2^k}+1}{4^{2^k}-1} - \frac{1}{4^{2^k}-1} = \frac{1}{2^{2^k}-1} - \frac{1}{4^{2^k}-1} = \frac{1}{4^{2^{k-1}}-1} - \frac{1}{4^{2^k}-1}.$$

   Therefore, the sum telescopes as

   $$\left(\frac{1}{4^{2^{-1}}-1} - \frac{1}{4^{2^0}-1}\right) + \left(\frac{1}{4^{2^0}-1} - \frac{1}{4^{2^1}-1}\right) + \left(\frac{1}{4^{2^1}-1} - \frac{1}{4^{2^2}-1}\right) + \cdots$$

   and evaluates to $1/(4^{2^{-1}}-1) = 1$.

4. What is the probability that in a randomly chosen arrangement of the numbers and letters in "HMMT2005," one can read either "HMMT" or "2005" from left to right? (For example, in "5HM0M20T," one can read "HMMT.")

   Solution: $\boxed{23/144}$

To read "HMMT," there are $\binom{8}{4}$ ways to place the letters, and $\frac{4!}{2}$ ways to place the numbers. Similarly, there are $\binom{8}{4}\frac{4!}{2}$ arrangements where one can read "2005." The number of arrangements in which one can read both is just $\binom{8}{4}$. The total number of arrangements is $\frac{8!}{4}$, thus the answer is

$$\frac{\binom{8}{4}\frac{4!}{2}+\binom{8}{4}\frac{4!}{2}-\binom{8}{4}}{\frac{8!}{4}}=\binom{8}{4}\frac{4}{8!}\cdot 23=\frac{23}{144}.$$

5. For how many integers n between 1 and 2005, inclusive, is $2\cdot 6\cdot 10\cdots(4n-2)$ divisible by $n!$?

   Solution: $\boxed{2005}$

   Note that

$$\begin{aligned} 2\cdot 6\cdot 10\cdots(4n-2) &= 2^n\cdot 1\cdot 3\cdot 5\cdots(2n-1) \\ &= 2^n\cdot\frac{1\cdot 2\cdot 3\cdots 2n}{2\cdot 4\cdot 6\cdots 2n} \\ &= \frac{1\cdot 2\cdot 3\cdots 2n}{1\cdot 2\cdot 3\cdots n}, \end{aligned}$$

   that is, it is just $(2n)!/n!$. Therefore, since $(2n)!/(n!)^2=\binom{2n}{n}$ is always an integer, the answer is 2005.

6. Let $m\circ n=(m+n)/(mn+4)$. Compute $((\cdots((2005\circ 2004)\circ 2003)\circ\cdots\circ 1)\circ 0)$.

   Solution: $\boxed{1/12}$

   Note that $m\circ 2=(m+2)/(2m+4)=\frac{1}{2}$, so the quantity we wish to find is just $(\frac{1}{2}\circ 1)\circ 0=\frac{1}{3}\circ 0=1/12$.

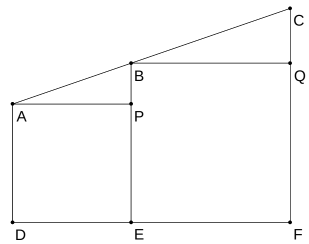
7. Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

   Solution: $\boxed{\sqrt{21}}$

   If A, B, and C are the tops of the heads of three successive people and D, E, and F are their respective feet, let P be the foot of the perpendicular from A to BE and let Q be the foot of the perpendicular from B to CF. Then, by equal angles, $\triangle ABP\sim\triangle BCQ$, so

$$\frac{CF}{BE}=\frac{CF}{BQ}=\frac{CQ}{BQ}+1=\frac{BP}{AP}+1=\frac{BE}{AP}=\frac{BE}{AD}.$$

   Therefore the heights of successive people are in geometric progression. Hence, the heights of all five people are in geometric progression, so the middle height is $\sqrt{3\cdot 7}=\sqrt{21}$ feet.



8. Let $ABCD$ be a convex quadrilateral inscribed in a circle with shortest side AB. The ratio $[BCD]/[ABD]$ is an integer (where $[XYZ]$ denotes the area of triangle XYZ.) If the lengths of AB, BC, CD, and DA are distinct integers no greater than 10, find the largest possible value of AB.

   Solution: $\boxed{5}$

   Note that
$$\frac{[BCD]}{[ABD]} = \frac{\frac{1}{2}BC \cdot CD \cdot \sin C}{\frac{1}{2}DA \cdot AB \cdot \sin A} = \frac{BC \cdot CD}{DA \cdot AB}$$
   since $\angle A$ and $\angle C$ are supplementary. If $AB \geq 6$, it is easy to check that no assignment of lengths to the four sides yields an integer ratio, but if $AB = 5$, we can let $BC = 10$, $CD = 9$, and $DA = 6$ for a ratio of 3. The maximum value for AB is therefore 5.

9. Farmer Bill's 1000 animals — ducks, cows, and rabbits — are standing in a circle. In order to feel safe, every duck must either be standing next to at least one cow or between two rabbits. If there are 600 ducks, what is the least number of cows there can be for this to be possible?

   Solution: $\boxed{201}$

   Suppose Bill has r rabbits and c cows. At most $r-1$ ducks can be between two rabbits: each rabbit can serve up to two such ducks, so at most $2r/2 = r$ ducks will each be served by two rabbits, but we cannot have equality, since this would require alternating between rabbits and ducks all the way around the circle, contradicting the fact that more than half the animals are ducks. Also, at most $2c$ ducks can each be adjacent to a cow. So we need $600 \leq r - 1 + 2c = (400 - c) - 1 + 2c$, giving $c \geq 201$. Conversely, an arrangement with 201 cows is possible:
$$\underbrace{RDRDR \cdots DR}_{199\ R,\ 198\ D}\underbrace{DCD\ DCD\ DCD\ \cdots\ DCD}_{201\ C,\ 402\ D}.$$
   So 201 is the answer.

10. You are given a set of cards labeled from 1 to 100. You wish to make piles of three cards such that in any pile, the number on one of the cards is the product of the numbers on the other two cards. However, no card can be in more than one pile. What is the maximum number of piles you can form at once?

    Solution: $\boxed{8}$

    Certainly, the two factors in any pile cannot both be at least 10, since then the product would be at least $10 \times 11 > 100$. Also, the number 1 can not appear in any pile, since then the other two cards in the pile would have to be the same. So each pile must use one of the numbers $2, 3, \ldots, 9$ as one of the factors, meaning we have at most 8 piles. Conversely, it is easy to construct a set of 8 such piles, for example:

$$\{9, 11, 99\} \quad \{8, 12, 96\} \quad \{7, 13, 91\} \quad \{6, 14, 84\}$$

$$\{5, 15, 75\} \quad \{4, 16, 64\} \quad \{3, 17, 51\} \quad \{2, 18, 36\}$$

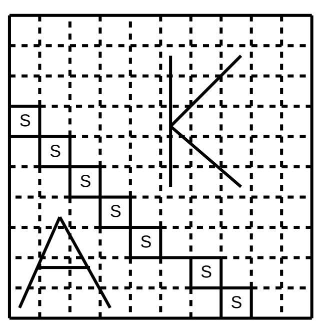
11. The Dingoberry Farm is a 10 mile by 10 mile square, broken up into 1 mile by 1 mile patches. Each patch is farmed either by Farmer Keith or by Farmer Ann. Whenever Ann farms a patch, she also farms all the patches due west of it and all the patches due south of it. Ann puts up a scarecrow on each of her patches that is adjacent to exactly two of Keith's patches (and nowhere else). If Ann farms a total of 30 patches, what is the largest number of scarecrows she could put up?

Solution: $\boxed{7}$

Whenever Ann farms a patch P, she also farms all the patches due west of P and due south of P. So, the only way she can put a scarecrow on P is if Keith farms the patch immediately north of P and the patch immediately east of P, in which case Ann cannot farm any of the patches due north of P or due east of P. That is, Ann can only put a scarecrow on P if it is the easternmost patch she farms in its east-west row, and the northernmost in its north-south column. In particular, all of her scarecrow patches are in different rows and columns. Suppose that she puts up n scarecrows. The farthest south of these must be in the 10th row or above, so she farms at least 1 patch in that column; the second-farthest south must be in the 9th row above, so she farms at least 2 patches in that column; the third-farthest south must be in the 8th row or above, so she farms at least 3 patches in that column, and so forth, for a total of at least

$$1 + 2 + \cdots + n = n(n + 1)/2$$

patches. If Ann farms a total of $30 < 8 \cdot 9/2$ patches, then we have $n < 8$. On the other hand, $n = 7$ scarecrows are possible, as shown:



12. Two vertices of a cube are given in space. The locus of points that could be a third vertex of the cube is the union of n circles. Find n.

Solution: $\boxed{10}$

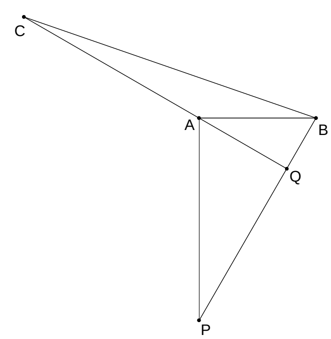
Let the distance between the two given vertices be 1. If the two given vertices are adjacent, then the other vertices lie on four circles, two of radius 1 and two of radius $\sqrt{2}$. If the two vertices are separated by a diagonal of a face of the cube, then the locus of possible vertices adjacent to both of them is a circle of radius $\frac{1}{2}$, the locus of possible vertices adjacent to exactly one of them is two circles of radius $\frac{\sqrt{2}}{2}$, and the locus of possible vertices adjacent to neither of them is a circle of radius $\frac{\sqrt{3}}{2}$. If the two given vertices are separated by a long diagonal, then each of the other vertices lie on one of two circles of radius $\frac{\sqrt{2}}{3}$, for a total of 10 circles.

13. Triangle ABC has $AB = 1$, $BC = \sqrt{7}$, and $CA = \sqrt{3}$. Let ℓ_1 be the line through A perpendicular to AB, ℓ_2 the line through B perpendicular to AC, and P the point of intersection of ℓ_1 and ℓ_2. Find PC.

    Solution: $\boxed{3}$

    By the Law of Cosines, $\angle BAC = \cos^{-1}\frac{3+1-7}{2\sqrt{3}} = \cos^{-1}(-\frac{\sqrt{3}}{2}) = 150°$. If we let Q be the intersection of ℓ_2 and AC, we notice that $\angle QBA = 90° - \angle QAB = 90° - 30° = 60°$. It follows that triangle ABP is a 30-60-90 triangle and thus $PB = 2$ and $PA = \sqrt{3}$. Finally, we have $\angle PAC = 360° - (90° + 150°) = 120°$, and

$$PC = (PA^2 + AC^2 - 2PA \cdot AC \cos 120°)^{1/2} = (3 + 3 + 3)^{1/2} = 3.$$



14. Three noncollinear points and a line ℓ are given in the plane. Suppose no two of the points lie on a line parallel to ℓ (or ℓ itself). There are exactly n lines perpendicular to ℓ with the following property: the three circles with centers at the given points and tangent to the line all concur at some point. Find all possible values of n.

    Solution: $\boxed{1}$

    The condition for the line is that each of the three points lies at an equal distance from the line as from some fixed point; in other words, the line is the directrix of a parabola containing the three points. Three noncollinear points in the coordinate plane determine a quadratic polynomial in x unless two of the points have the same x-coordinate. Therefore, given the direction of the directrix, three noncollinear points determine a parabola, unless two of the points lie on a line perpendicular to the directrix. This case is ruled out by the given condition, so the answer is 1.

15. Let S be the set of lattice points inside the circle $x^2 + y^2 = 11$. Let M be the greatest area of any triangle with vertices in S. How many triangles with vertices in S have area M?

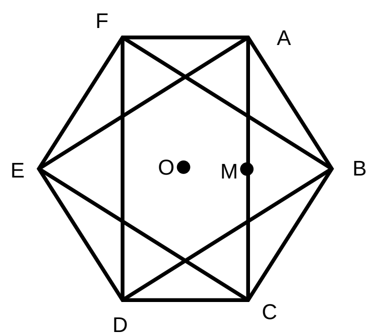
    Solution: $\boxed{16}$

    The boundary of the convex hull of S consists of points with (x, y) or $(y, x) = (0, \pm 3)$, $(\pm 1, \pm 3)$, and $(\pm 2, \pm 2)$. For any triangle T with vertices in S, we can increase its area by moving a vertex not on the boundary to some point on the boundary. Thus, if T has area M, its vertices are all on the boundary of S. The next step is to see (either by inspection or by noting that T has area no larger than that of an equilateral triangle inscribed in a circle of radius $\sqrt{10}$, which has area less than 13) that $M = 12$. There are 16 triangles with area 12, all congruent to one of the following three: vertices $(2, 2)$, $(1, -3)$, and $(-3, 1)$; vertices $(3, -1)$, $(-3, -1)$, and $(1, 3)$; or vertices $(3, -1)$, $(-3, -1)$, and $(0, 3)$.

16. A regular octahedron has a side length of 1. What is the distance between two opposite faces?

Solution: $\boxed{\sqrt{6}/3}$

Imagine orienting the octahedron so that the two opposite faces are horizontal. Project onto a horizontal plane; these two faces are congruent equilateral triangles which (when projected) have the same center and opposite orientations. Hence, the vertices of the octahedron project to the vertices of a regular hexagon $ABCDEF$.



Let O be the center of the hexagon and M the midpoint of AC. Now ABM is a 30-60-90 triangle, so $AB = AM/(\sqrt{3}/2) = (1/2)/(\sqrt{3}/2) = \sqrt{3}/3$. If we let d denote the desired vertical distance between the opposite faces (which project to ACE and BDF), then by the Pythagorean Theorem, $AB^2 + d^2 = 1^2$, so $d = \sqrt{1 - AB^2} = \sqrt{6}/3$.

17. Compute

$$2\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2\cdots}}}}.$$

Solution: $\boxed{2^{e-1}}$

Taking the base 2 logarithm of the expression gives

$$1 + \frac{1}{2}\left(1 + \frac{1}{3}\left(1 + \frac{1}{4}(1 + \cdots)\right)\right) = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e - 1.$$

Therefore the expression is just 2^{e-1}.

18. If a, b, and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a, b, and c?

Solution: $\boxed{1/4}$

Let d be a fourth random variable, also chosen uniformly from $[0, 1]$. For fixed a, b, and c, the probability that $d < \min\{a, b, c\}$ is evidently equal to $\min\{a, b, c\}$. Hence, if we average over all choices of a, b, c, the average value of $\min\{a,b,c\}$ is equal to the probability that, when a, b, c, and d are independently randomly chosen, $d < \min\{a,b,c\}$, i.e., that d is the smallest of the four variables. On the other hand, by symmetry, the probability that d is the smallest of the four is simply equal to $1/4$, so that is our answer.

19. Regular tetrahedron $ABCD$ is projected onto a plane sending A, B, C, and D to A', B', C', and D' respectively. Suppose $A'B'C'D'$ is a convex quadrilateral with $A'B' = A'D'$ and $C'B' = C'D'$, and suppose that the area of $A'B'C'D' = 4$. Given these conditions, the set of possible lengths of AB consists of all real numbers in the interval $[a, b)$. Compute b.

    Solution: $\boxed{2\sqrt[4]{6}}$

    The value of b occurs when the quadrilateral $A'B'C'D'$ degenerates to an isosceles triangle. This occurs when the altitude from A to BCD is parallel to the plane. Let $s = AB$. Then the altitude from A intersects the center E of face BCD. Since $EB = \frac{s}{\sqrt{3}}$, it follows that $A'C' = AE = \sqrt{s^2 - \frac{s^2}{3}} = \frac{s\sqrt{6}}{3}$. Then since BD is parallel to the plane, $B'D' = s$. Then the area of $A'B'C'D'$ is $4 = \frac{1}{2} \cdot \frac{s^2\sqrt{6}}{3}$, implying $s^2 = 4\sqrt{6}$, or $s = 2\sqrt[4]{6}$.

20. If n is a positive integer, let $s(n)$ denote the sum of the digits of n. We say that n is *zesty* if there exist positive integers x and y greater than 1 such that $xy = n$ and $s(x)s(y) = s(n)$. How many zesty two-digit numbers are there?

    Solution: $\boxed{34}$

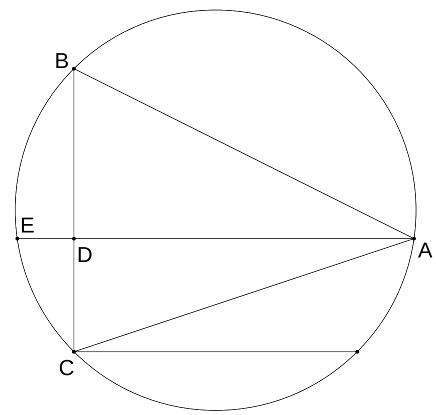
    Let n be a zesty two-digit number, and let x and y be as in the problem statement. Clearly if both x and y are one-digit numbers, then $s(x)s(y) = n \neq s(n)$. Thus either x is a two-digit number or y is. Assume without loss of generality that it is x. If $x = 10a + b$, $1 \leq a \leq 9$ and $0 \leq b \leq 9$, then $n = 10ay + by$. If both ay and by are less than 10, then $s(n) = ay + by$, but if either is at least 10, then $s(n) < ay + by$. It follows that the two digits of n share a common factor greater than 1, namely y. It is now easy to count the zesty two-digit numbers by first digit starting with 2; there are a total of $5 + 4 + 5 + 2 + 7 + 2 + 5 + 4 = 34$.

21. In triangle ABC with altitude AD, $\angle BAC = 45°$, $DB = 3$, and $CD = 2$. Find the area of triangle ABC.

    Solution: $\boxed{15}$

    Suppose first that D lies between B and C. Let ABC be inscribed in circle ω, and extend AD to intersect ω again at E. Note that A subtends a quarter of the circle, so in particular, the chord through C perpendicular to BC and parallel to AD has length $BC = 5$. Therefore, $AD = 5 + DE$. By power of a point, $6 = BD \cdot DC = AD \cdot DE = AD^2 - 5AD$, implying $AD = 6$, so the area of ABC is $\frac{1}{2}BC \cdot AD = 15$.

    If D does not lie between B and C, then $BC = 1$, so A lies on a circle of radius $\sqrt{2}/2$ through B and C. But then it is easy to check that the perpendicular to BC through D cannot intersect the circle, a contradiction.



22. Find
$$\{\ln(1+e)\} + \{\ln(1+e^2)\} + \{\ln(1+e^4)\} + \{\ln(1+e^8)\} + \cdots,$$
where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

Solution: $\boxed{1 - \ln(e-1)}$

Since $\ln(1+e^{2^k})$ is just larger than 2^k, its fractional part is $\ln(1+e^{2^k}) - \ln e^{2^k} = \ln(1+e^{-2^k})$. But now notice that
$$\prod_{k=0}^{n}(1+x^{2^k}) = 1 + x + x^2 + \cdots + x^{2^{n+1}-1}.$$
(This is easily proven by induction or by noting that every nonnegative integer less than 2^{n+1} has a unique $(n+1)$-bit binary expansion.) If $|x| < 1$, this product converges to $\frac{1}{1-x}$ as n goes to infinity. Therefore,
$$\sum_{k=0}^{\infty} \ln(1+e^{-2^k}) = \ln \prod_{k=0}^{\infty}(1+(e^{-1})^{2^k}) = \ln \frac{1}{1-e^{-1}} = \ln \frac{e}{e-1} = 1 - \ln(e-1).$$

23. The sides of a regular hexagon are trisected, resulting in 18 points, including vertices. These points, starting with a vertex, are numbered clockwise as $A_1, A_2, \ldots, A_{18}$. The line segment A_kA_{k+4} is drawn for $k = 1, 4, 7, 10, 13, 16$, where indices are taken modulo 18. These segments define a region containing the center of the hexagon. Find the ratio of the area of this region to the area of the large hexagon.

Solution: $\boxed{9/13}$

Let us assume all sides are of side length 3. Consider the triangle $A_1A_4A_5$. Let P be the point of intersection of A_1A_5 with A_4A_8. This is a vertex of the inner hexagon. Then $\angle A_4A_1A_5 = \angle A_5A_4P$, by symmetry. It follows that $A_1A_4A_5 \sim A_4PA_5$. Also, $\angle A_1A_4A_5 = 120°$, so by the Law of Cosines $A_1A_5 = \sqrt{13}$. It follows that $PA_5 = (A_4A_5) \cdot (A_4A_5)/(A_1A_5) = 1/\sqrt{13}$. Let Q be the intersection of A_1A_5 and $A_{16}A_2$. By similar reasoning, $A_1Q = 3/\sqrt{13}$, so $PQ = A_1A_5 - A_1Q - PA_5 = 9/\sqrt{13}$. By symmetry, the inner region is a regular hexagon with side length $9/\sqrt{13}$. Hence the ratio of the area of the smaller to larger hexagon is $(3/\sqrt{13})^2 = 9/13$.

24. In the base 10 arithmetic problem $HMMT + GUTS = ROUND$, each distinct letter represents a different digit, and leading zeroes are not allowed. What is the maximum possible value of $ROUND$?

Solution: $\boxed{16352}$

Clearly $R = 1$, and from the hundreds column, $M = 0$ or 9. Since $H + G = 9 + O$ or $10 + O$, it is easy to see that O can be at most 7, in which case H and G must be 8 and 9, so $M = 0$. But because of the tens column, we must have $S + T \geq 10$, and in fact since D cannot be 0 or 1, $S + T \geq 12$, which is impossible given the remaining choices. Therefore, O is at most 6.

Suppose $O = 6$ and $M = 9$. Then we must have H and G be 7 and 8. With the remaining digits 0, 2, 3, 4, and 5, we must have in the ones column that T and S are 2 and 3, which leaves no possibility for N. If instead $M = 0$, then H and G are 7 and 9. Since again $S + T \geq 12$ and $N = T + 1$, the only possibility is $S = 8$, $T = 4$, and $N = 5$, giving $ROUND = 16352 = 7004 + 9348 = 9004 + 7348$.

25. An ant starts at one vertex of a tetrahedron. Each minute it walks along a random edge to an adjacent vertex. What is the probability that after one hour the ant winds up at the same vertex it started at?

Solution: $\boxed{(3^{59} + 1)/(4 \cdot 3^{59})}$

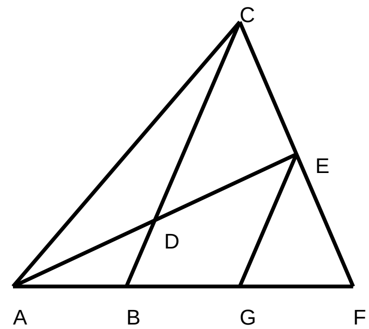
Let p_n be the probability that the ant is at the original vertex after n minutes; then $p_0 = 1$. The chance that the ant is at each of the other three vertices after n minutes is $\frac{1}{3}(1-p_n)$. Since the ant can only walk to the original vertex from one of the three others, and at each there is a $\frac{1}{3}$ probability of doing so, we have that $p_{n+1} = \frac{1}{3}(1 - p_n)$. Let $q_n = p_n - \frac{1}{4}$. Substituting this into the recurrence, we find that $q_{n+1} = \frac{1}{4} + \frac{1}{3}(-q_n - \frac{3}{4}) = -\frac{1}{3}q_n$. Since $q_0 = \frac{3}{4}$, $q_n = \frac{3}{4} \cdot \left(-\frac{1}{3}\right)^n$. In particular, this implies that

$$p_{60} = \frac{1}{4} + q_{60} = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3^{60}} = \frac{3^{59} + 1}{4 \cdot 3^{59}}.$$

26. In triangle ABC, $AC = 3AB$. Let AD bisect angle A with D lying on BC, and let E be the foot of the perpendicular from C to AD. Find $[ABD]/[CDE]$. (Here, $[XYZ]$ denotes the area of triangle XYZ).

Solution: $\boxed{1/3}$

By the Angle Bisector Theorem, $DC/DB = AC/AB = 3$. We will show that $AD = DE$. Let CE intersect AB at F. Then since AE bisects angle A, $AF = AC = 3AB$, and $EF = EC$. Let G be the midpoint of BF. Then $BG = GF$, so $GE \parallel BC$. But then since B is the midpoint of AG, D must be the midpoint of AE, as desired. Then $[ABD]/[CDE] = (AD \cdot BD)/(ED \cdot CD) = 1/3$.



27. In a chess-playing club, some of the players take lessons from other players. It is possible (but not necessary) for two players both to take lessons from each other. It so happens that for any three distinct members of the club, A, B, and C, exactly one of the following three statements is true: A takes lessons from B; B takes lessons from C; C takes lessons from A. What is the largest number of players there can be?

Solution: $\boxed{4}$

If P, Q, R, S, and T are any five distinct players, then consider all pairs $A, B \in \{P, Q, R, S, T\}$ such that A takes lessons from B. Each pair contributes to exactly three triples (A, B, C) (one for each of the choices of C distinct from A and B); three triples (C, A, B); and three triples (B, C, A). On the other hand, there are $5 \times 4 \times 3 = 60$ ordered triples of distinct players among these five, and each includes exactly one of our lesson-taking pairs. That means that there are $60/9$ such pairs. But this number isn't an integer, so there cannot be five distinct people in the club.

On the other hand, there can be four people, P, Q, R, and S: let P and Q both take lessons from each other, and let R and S both take lessons from each other; it is easy to check that this meets the conditions. Thus the maximum number of players is 4.

28. There are three pairs of real numbers (x_1, y_1), (x_2, y_2), and (x_3, y_3) that satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$. Compute $\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_2}\right)\left(1 - \frac{x_3}{y_3}\right)$.

Solution: $\boxed{1/1002}$

By the given, $2004(x^3 - 3xy^2) - 2005(y^3 - 3x^2y) = 0$. Dividing both sides by y^3 and setting $t = \frac{x}{y}$ yields $2004(t^3 - 3t) - 2005(1 - 3t^2) = 0$. A quick check shows that this cubic has three real roots. Since the three roots are precisely $\frac{x_1}{y_1}$, $\frac{x_2}{y_2}$, and $\frac{x_3}{y_3}$, we must have $2004(t^3 - 3t) - 2005(1 - 3t^2) = 2004\left(t - \frac{x_1}{y_1}\right)\left(t - \frac{x_2}{y_2}\right)\left(t - \frac{x_3}{y_3}\right)$. Therefore,

$$\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_2}\right)\left(1 - \frac{x_3}{y_3}\right) = \frac{2004(1^3 - 3(1)) - 2005(1 - 3(1)^2)}{2004} = \frac{1}{1002}.$$

29. Let $n > 0$ be an integer. Each face of a regular tetrahedron is painted in one of n colors (the faces are not necessarily painted different colors.) Suppose there are n^3 possible colorings, where rotations, but not reflections, of the same coloring are considered the same. Find all possible values of n.

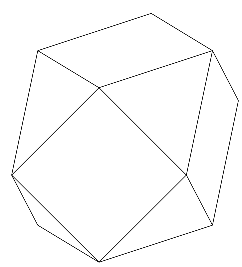
Solution: $\boxed{1, 11}$

We count the possible number of colorings. If four colors are used, there are two different colorings that are mirror images of each other, for a total of $2\binom{n}{4}$ colorings. If three colors are used, we choose one color to use twice (which determines the coloring), for a total of $3\binom{n}{3}$ colorings. If two colors are used, we can either choose one of those colors and color three faces with it, or we can color two faces each color, for a total of $3\binom{n}{2}$ colorings. Finally, we can also use only one color, for $\binom{n}{1}$ colorings. This gives a total of

$$2\binom{n}{4} + 3\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1} = \frac{1}{12}n^2(n^2+11)$$

colorings. Setting this equal to n^3, we get the equation $n^2(n^2+11) = 12n^3$, or equivalently $n^2(n-1)(n-11) = 0$, giving the answers 1 and 11.

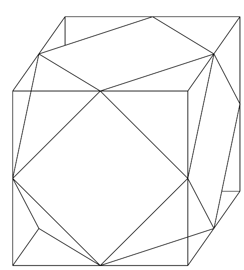
30. A cuboctahedron is a polyhedron whose faces are squares and equilateral triangles such that two squares and two triangles alternate around each vertex, as shown.



What is the volume of a cuboctahedron of side length 1?

Solution: $\boxed{5\sqrt{2}/3}$

We can construct a cube such that the vertices of the cuboctahedron are the midpoints of the edges of the cube.

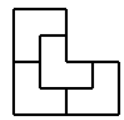


Let s be the side length of this cube. Now, the cuboctahedron is obtained from the cube by cutting a tetrahedron from each corner. Each such tetrahedron has a base in the form of an isosceles right triangle of area $(s/2)^2/2$ and height $s/2$ for a volume of $(s/2)^3/6$. The total volume of the cuboctahedron is therefore

$$s^3 - 8 \cdot (s/2)^3/6 = 5s^3/6.$$

Now, the side of the cuboctahedron is the hypotenuse of an isosceles right triangle of leg $s/2$; thus $1 = (s/2)\sqrt{2}$, giving $s = \sqrt{2}$, so the volume of the cuboctahedron is $5\sqrt{2}/3$.

31. The L shape made by adjoining three congruent squares can be subdivided into four smaller L shapes.



Each of these can in turn be subdivided, and so forth. If we perform 2005 successive subdivisions, how many of the 4^{2005} L's left at the end will be in the same orientation as the original one?

Solution: $\boxed{4^{2004} + 2^{2004}}$

After n successive subdivisions, let a_n be the number of small L's in the same orientation as the original one; let b_n be the number of small L's that have this orientation rotated counterclockwise 90°; let c_n be the number of small L's that are rotated 180°; and let d_n be the number of small L's that are rotated 270°. When an L is subdivided, it produces two smaller L's of the same orientation, one of each of the neighboring orientations, and none of the opposite orientation. Therefore,

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (d_n + 2a_n + b_n,\ a_n + 2b_n + c_n,\ b_n + 2c_n + d_n,\ c_n + 2d_n + a_n).$$

It is now straightforward to show by induction that

$$(a_n, b_n, c_n, d_n) = (4^{n-1} + 2^{n-1},\ 4^{n-1},\ 4^{n-1} - 2^{n-1},\ 4^{n-1})$$

for each $n \geq 1$. In particular, our desired answer is $a_{2005} = 4^{2004} + 2^{2004}$.

32. Let $a_1 = 3$, and for $n \geq 1$, let $a_{n+1} = (n+1)a_n - n$. Find the smallest $m \geq 2005$ such that $a_{m+1} - 1 \mid a_m^2 - 1$.

Solution: $\boxed{2010}$

We will show that $a_n = 2 \cdot n! + 1$ by induction. Indeed, the claim is obvious for $n = 1$, and $(n+1)(2 \cdot n! + 1) - n = 2 \cdot (n+1)! + 1$. Then we wish to find $m \geq 2005$ such that $2(m+1)! \mid 4(m!)^2 + 4m!$, or dividing by $2 \cdot m!$, we want $m + 1 \mid 2(m! + 1)$. Suppose $m+1$ is composite. Then it has a proper divisor $d > 2$, and since $d \mid m!$, we must have $d \mid 2$, which is impossible. Therefore, $m+1$ must be prime, and if this is the case, then $m+1 \mid m!+1$ by Wilson's Theorem. Therefore, since the smallest prime greater than 2005 is 2011, the smallest possible value of m is 2010.

33. Triangle ABC has incircle ω which touches AB at C_1, BC at A_1, and CA at B_1. Let A_2 be the reflection of A_1 over the midpoint of BC, and define B_2 and C_2 similarly. Let A_3 be the intersection of AA_2 with ω that is closer to A, and define B_3 and C_3 similarly. If $AB = 9$, $BC = 10$, and $CA = 13$, find $[A_3B_3C_3]/[ABC]$. (Here $[XYZ]$ denotes the area of triangle XYZ.)

Solution: $\boxed{14/65}$

Notice that A_2 is the point of tangency of the excircle opposite A to BC. Therefore, by considering the homothety centered at A taking the excircle to the incircle, we notice that A_3 is the intersection of ω and the tangent line parallel to BC. It follows that

$A_1B_1C_1$ is congruent to $A_3B_3C_3$ by reflecting through the center of ω. We therefore need only find $[A_1B_1C_1]/[ABC]$. Since

$$\frac{[A_1BC_1]}{[ABC]} = \frac{A_1B \cdot BC_1}{AB \cdot BC} = \frac{((9+10-13)/2)^2}{9 \cdot 10} = \frac{1}{10},$$

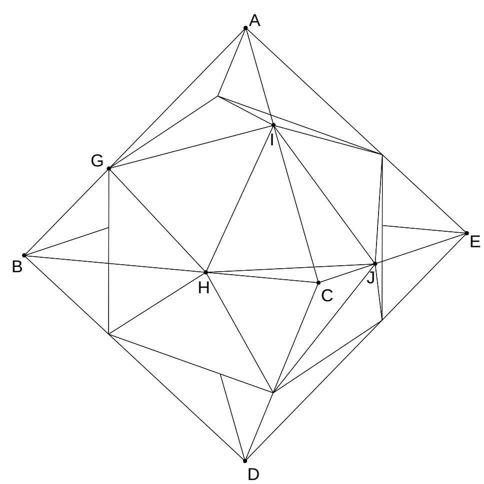
and likewise $[A_1B_1C]/[ABC] = 49/130$ and $[AB_1C_1]/[ABC] = 4/13$, we get that

$$\frac{[A_3B_3C_3]}{[ABC]} = 1 - \frac{1}{10} - \frac{49}{130} - \frac{4}{13} = \frac{14}{65}.$$

34. A regular octahedron $ABCDEF$ is given such that AD, BE, and CF are perpendicular. Let G, H, and I lie on edges AB, BC, and CA respectively such that $\frac{AG}{GB} = \frac{BH}{HC} = \frac{CI}{IA} = \rho$. For some choice of $\rho > 1$, GH, HI, and IG are three edges of a regular icosahedron, eight of whose faces are inscribed in the faces of $ABCDEF$. Find ρ.

Solution: $\boxed{(1+\sqrt{5})/2}$

Let J lie on edge CE such that $\frac{EJ}{JC} = \rho$. Then we must have that HIJ is another face of the icosahedron, so in particular, $HI = HJ$. But since BC and CE are perpendicular, $HJ = HC\sqrt{2}$. By the Law of Cosines, $HI^2 = HC^2 + CI^2 - 2HC \cdot CI \cos 60^\circ = HC^2(1 + \rho^2 - \rho)$. Therefore, $2 = 1 + \rho^2 - \rho$, or $\rho^2 - \rho - 1 = 0$, giving $\rho = \frac{1+\sqrt{5}}{2}$.



35. Let $p = 2^{24036583} - 1$, the largest prime currently known. For how many positive integers c do the quadratics $\pm x^2 \pm px \pm c$ all have rational roots?

Solution: $\boxed{0}$

This is equivalent to both discriminants $p^2 \pm 4c$ being squares. In other words, p^2 must be the average of two squares a^2 and b^2. Note that a and b must have the same parity, and that $(\frac{a+b}{2})^2 + (\frac{a-b}{2})^2 = \frac{a^2+b^2}{2} = p^2$. Therefore, p must be the hypotenuse in a Pythagorean triple. Such triples are parametrized by $k(m^2 - n^2, 2mn, m^2 + n^2)$. But $p \equiv 3 \pmod 4$ and is therefore not the sum of two squares. This implies that p is not the hypotenuse of any Pythagorean triple, so the answer is 0.

36. One hundred people are in line to see a movie. Each person wants to sit in the front row, which contains one hundred seats, and each has a favorite seat, chosen randomly and independently. They enter the row one at a time from the far right. As they walk, if they reach their favorite seat, they sit, but to avoid stepping over people, if they encounter a person already seated, they sit to that person's right. If the seat furthest to the right is already taken, they sit in a different row. What is the most likely number of people that will get to sit in the first row?

Solution: $\boxed{10}$

Let $S(i)$ be the favorite seat of the ith person, counting from the right. Let $P(n)$ be the probability that *at least* n people get to sit. At least n people sit if and only if $S(1) \geq n$, $S(2) \geq n-1$, ..., $S(n) \geq 1$. This has probability:

$$P(n) = \frac{100-(n-1)}{100} \cdot \frac{100-(n-2)}{100} \cdots \frac{100}{100} = \frac{100!}{(100-n)! \cdot 100^n}.$$

The probability, $Q(n)$, that exactly n people sit is

$$P(n) - P(n+1) = \frac{100!}{(100-n)! \cdot 100^n} - \frac{100!}{(99-n)! \cdot 100^{n+1}} = \frac{100! \cdot n}{(100-n)! \cdot 100^{n+1}}.$$

Now,

$$\frac{Q(n)}{Q(n-1)} = \frac{100! \cdot n}{(100-n)! \cdot 100^{n+1}} \cdot \frac{(101-n)! \cdot 100^n}{100! \cdot (n-1)} = \frac{n(101-n)}{100(n-1)} = \frac{101n - n^2}{100n - 100},$$

which is greater than 1 exactly when $n^2 - n - 100 < 0$, that is, for $n \leq 10$. Therefore, the maximum value of $Q(n)$ occurs for $n = 10$.

37. Let $a_1, a_2, \ldots, a_{2005}$ be real numbers such that

$$\begin{array}{ccccccccccc} a_1 \cdot 1 & + & a_2 \cdot 2 & + & a_3 \cdot 3 & + & \cdots & + & a_{2005} \cdot 2005 & = & 0 \\ a_1 \cdot 1^2 & + & a_2 \cdot 2^2 & + & a_3 \cdot 3^2 & + & \cdots & + & a_{2005} \cdot 2005^2 & = & 0 \\ a_1 \cdot 1^3 & + & a_2 \cdot 2^3 & + & a_3 \cdot 3^3 & + & \cdots & + & a_{2005} \cdot 2005^3 & = & 0 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_1 \cdot 1^{2004} & + & a_2 \cdot 2^{2004} & + & a_3 \cdot 3^{2004} & + & \cdots & + & a_{2005} \cdot 2005^{2004} & = & 0 \end{array}$$

and

$$a_1 \cdot 1^{2005} + a_2 \cdot 2^{2005} + a_3 \cdot 3^{2005} + \cdots + a_{2005} \cdot 2005^{2005} = 1.$$

What is the value of a_1?

Solution: $\boxed{1/2004!}$

The polynomial $p(x) = x(x-2)(x-3)\cdots(x-2005)/2004!$ has zero constant term, has the numbers $2, 3, \ldots, 2005$ as roots, and satisfies $p(1) = 1$. Multiplying the nth equation by the coefficient of x^n in the polynomial $p(x)$ and summing over all n gives

$$a_1 p(1) + a_2 p(2) + a_3 p(3) + \cdots + a_{2005} p(2005) = 1/2004!$$

(since the leading coefficient is $1/2004!$). The left side just reduces to a_1, so $1/2004!$ is the answer.

38. In how many ways can the set of ordered pairs of integers be colored red and blue such that for all a and b, the points (a, b), $(-1-b, a+1)$, and $(1-b, a-1)$ are all the same color?

Solution: $\boxed{16}$

Let φ_1 and φ_2 be $90°$ counterclockwise rotations about $(-1, 0)$ and $(1, 0)$, respectively. Then $\varphi_1(a, b) = (-1-b, a+1)$, and $\varphi_2(a, b) = (1-b, a-1)$. Therefore, the possible colorings are precisely those preserved under these rotations. Since $\varphi_1(1, 0) = (-1, 2)$, the colorings must also be preserved under $90°$ rotations about $(-1, 2)$. Similarly, one can show that they must be preserved under rotations about any point (x, y), where x is odd and y is even. Decompose the lattice points as follows:

$$\begin{aligned} L_1 &= \{(x, y) \mid x + y \equiv 0 \pmod 2\} \\ L_2 &= \{(x, y) \mid x \equiv y - 1 \equiv 0 \pmod 2\} \\ L_3 &= \{(x, y) \mid x + y - 1 \equiv y - x + 1 \equiv 0 \pmod 4\} \\ L_4 &= \{(x, y) \mid x + y + 1 \equiv y - x - 1 \equiv 0 \pmod 4\} \end{aligned}$$

Within any of these sublattices, any point can be brought to any other through appropriate rotations, but no point can be brought to any point in a different sublattice. It follows that every sublattice must be colored in one color, but that different sublattices can be colored differently. Since each of these sublattices can be colored in one of two colors, there are $2^4 = 16$ possible colorings.

1	2	1	2	1	2	1
4	1	3	1	4	1	3
1	2	1	2	1	2	1
3	1	4	1	3	1	4
1	2	1	2	1	2	1
4	1	3	1	4	1	3
1	2	1	2	1	2	1

39. How many regions of the plane are bounded by the graph of

$$x^6 - x^5 + 3x^4y^2 + 10x^3y^2 + 3x^2y^4 - 5xy^4 + y^6 = 0?$$

Solution: $\boxed{5}$

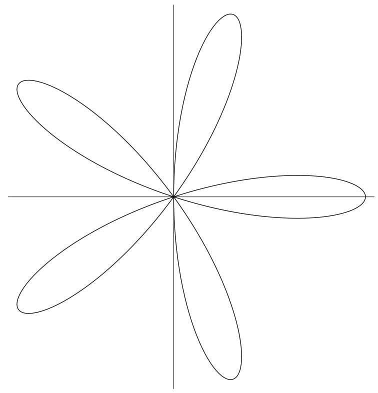
The left-hand side decomposes as

$$(x^6 + 3x^4y^2 + 3x^2y^4 + y^6) - (x^5 - 10x^3y^2 + 5xy^4) = (x^2 + y^2)^3 - (x^5 - 10x^3y^2 + 5xy^4).$$

Now, note that

$$(x + iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5,$$

so that our function is just $(x^2 + y^2)^3 - \Re((x + iy)^5)$. Switching to polar coordinates, this is $r^6 - \Re(r^5(\cos\theta + i\sin\theta)^5) = r^6 - r^5\cos 5\theta$ by de Moivre's rule. The graph of our function is then the graph of $r^6 - r^5\cos 5\theta = 0$, or, more suitably, of $r = \cos 5\theta$. This is a five-petal rose, so the answer is 5.



40. In a town of n people, a governing council is elected as follows: each person casts one vote for some person in the town, and anyone that receives at least five votes is elected to council. Let $c(n)$ denote the average number of people elected to council if everyone votes randomly. Find $\lim_{n\to\infty} c(n)/n$.

Solution: $\boxed{1 - 65/24e}$

Let $c_k(n)$ denote the expected number of people that will receive exactly k votes. We will show that $\lim_{n\to\infty} c_k(n)/n = 1/(e \cdot k!)$. The probability that any given person receives exactly k votes, which is the same as the average proportion of people that receive exactly k votes, is

$$\binom{n}{k} \cdot \left(\frac{1}{n}\right)^k \cdot \left(\frac{n-1}{n}\right)^{n-k} = \left(\frac{n-1}{n}\right)^n \cdot \frac{n(n-1)\cdots(n-k+1)}{k! \cdot (n-1)^k}.$$

Taking the limit as $n \to \infty$ and noting that $\lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$ gives that the limit is $1/(e \cdot k!)$, as desired. Therefore, the limit of the average proportion of the town that receives at least five votes is

$$1 - \frac{1}{e}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right) = 1 - \frac{65}{24e}.$$

41. There are 42 stepping stones in a pond, arranged along a circle. You are standing on one of the stones. You would like to jump among the stones so that you move counterclockwise by either 1 stone or 7 stones at each jump. Moreover, you would like to do this in such a way that you visit each stone (except for the starting spot) exactly once before returning to your initial stone for the first time. In how many ways can you do this?

Solution: $\boxed{63}$

Number the stones $0, 1, \ldots, 41$, treating the numbers as values modulo 42, and let r_n be the length of your jump from stone n. If you jump from stone n to $n+7$, then you cannot jump from stone $n+6$ to $n+7$ and so must jump from $n+6$ to $n+13$. That is, if $r_n = 7$, then $r_{n+6} = 7$ also. It follows that the 7 values $r_n, r_{n+6}, r_{n+12}, \ldots, r_{n+36}$ are all equal: if one of them is 7, then by the preceding argument applied repeatedly, all of them must be 7, and otherwise all of them are 1. Now, for $n = 0, 1, 2, \ldots, 42$, let s_n be the stone you are on after n jumps. Then $s_{n+1} = s_n + r_{s_n}$, and we have $s_{n+1} = s_n + r_{s_n} \equiv s_n + 1 \pmod 6$. By induction, $s_{n+i} \equiv s_n + i \pmod 6$; in particular

$s_{n+6} \equiv s_n$, so $r_{s_n+6} = r_{s_n}$. That is, the sequence of jump lengths is periodic with period 6 and so is uniquely determined by the first 6 jumps. So this gives us at most $2^6 = 64$ possible sequences of jumps $r_{s_0}, r_{s_1}, \ldots, r_{s_{41}}$.

Now, the condition that you visit each stone exactly once before returning to the original stone just means that $s_0, s_1, \ldots, s_{41}$ are distinct and $s_{42} = s_0$. If all jumps are length 7, then $s_6 = s_0$, so this cannot happen. On the other hand, if the jumps are not all of length 7, then we claim $s_0, \ldots, s_{41}$ are indeed all distinct. Indeed, suppose $s_i = s_j$ for some $0 \le i < j < 42$. Since $s_j \equiv s_i + (j - i) \pmod 6$, we have $j \equiv i \pmod 6$, so $j - i = 6k$ for some k. Moreover, since the sequence of jump lengths has period 6, we have

$$s_{i+6} - s_i = s_{i+12} - s_{i+6} = \cdots = s_{i+6k} - s_{i+6(k-1)}.$$

Calling this common value l, we have $kl \equiv 0 \bmod 42$. But l is divisible by 6, and $j - i < 42 \Rightarrow k < 7$ means that k is not divisible by 7, so l must be. So l, the sum of six successive jump lengths, is divisible by 42. Hence the jumps must all be of length 7, as claimed.

This shows that, for the $64 - 1 = 63$ sequences of jumps that have period 6 and are not all of length 7, you do indeed reach every stone once before returning to the starting point.

42. In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid of boxes such that every row and column contains two balls of one color and one ball of the other color? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

Solution: $\boxed{5184}$

In each row or column, exactly one box is left empty. There are $4! = 24$ ways to choose the empty spots. Once that has been done, there are 6 ways to choose which two rows have 2 purple balls each. Now, assume without loss of generality that boxes $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$ are the empty ones, and that rows 1 and 2 have two purple balls each. Let A, B, C, and D denote the 2×2 squares in the top left, top right, bottom left, and bottom right corners, respectively (so A is formed by the first two rows and first two columns, etc.). Let a, b, c, and d denote the number of purple balls in A, B, C, and D, respectively. Then $0 \le a, d \le 2$, $a + b = 4$, and $b + d \le 4$, so $a \ge d$.

Now suppose we are given the numbers a and d, satisfying $0 \le d \le a \le 2$. Fortunately, the numbers of ways to color the balls in A, B, C, and D are independent of each other. For example, given $a = 1$ and $d = 0$, there are 2 ways to color A and 1 way to color D and, no matter how the coloring of A is done, there are always 2 ways to color B and 3 ways to color C. The numbers of ways to choose the colors of all the balls is as follows:

$a \backslash d$	0	1	2
0	$1 \cdot (1 \cdot 2) \cdot 1 = 2$	0	0
1	$2 \cdot (2 \cdot 3) \cdot 1 = 12$	$2 \cdot (1 \cdot 1) \cdot 2 = 4$	0
2	$1 \cdot (2 \cdot 2) \cdot 1 = 4$	$1 \cdot (3 \cdot 2) \cdot 2 = 12$	$1 \cdot (2 \cdot 1) \cdot 1 = 2$

In each square above, the four factors are the number of ways of arranging the balls in A, B, C, and D, respectively. Summing this over all pairs (a, d) satisfying $0 \le d \le a \le 2$ gives a total of 36. The answer is therefore $24 \cdot 6 \cdot 36 = 5184$.

43. Write down an integer N between 0 and 20 inclusive. If more than N teams write down N, your score is N; otherwise it is 0.

    Remark: Well, maybe you can get 1 point...

44. Write down a set S of positive integers, all greater than 1, such that for each $x \in S$, x is a proper divisor of $(P/x) + 1$, where P is the product of all the elements of S. Your score is $2n$, where $n = |S|$.

    Remark: This question was posed by Stefan Znám in 1972. Solutions exist for $|S| \geq 5$, but an explicit solution has been found only for $|S| \leq 13$. The simplest solution is $S = \{2, 3, 11, 23, 31\}$. A solution for $|S| = 7$ is given by $S = \{2, 3, 11, 17, 101, 149, 3109\}$. For more information, see `http://mathworld.wolfram.com/ZnamsProblem.html`.

45. A *binary word* is a finite sequence of 0's and 1's. A *square subword* is a subsequence consisting of two identical chunks next to each other. For example, the word 100101011 contains the square subwords 00, 0101 (twice), 1010, and 11.

    Find a long binary word containing a small number of square subwords. Specifically, write down a binary word of any length $n \leq 50$. Your score will be $\max\{0, n - s\}$, where s is the number of occurrences of square subwords. (That is, each different square subword will be counted according to the number of times it appears.)

    Remark: See `http://www.combinatorics.org/Volume_10/PDF/v10i1r12.pdf` for analysis of this problem. The maximum possible score is on the order of 25. In general, the minimum number of square subwords of a word of length n tends to roughly $0.551n$ as $n \to \infty$.