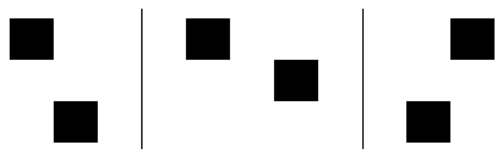
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round A

Disconnected Domino Rally [**175**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)*-tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Prove that for any two types of dominoes, there exists a rectangle that can be tiled by dominoes of either type.

2. [**25**] Suppose $0 < a \leq b$ and $4 \nmid mn$. Prove that the number of ways in which an $m \times n$ rectangle can be partitioned into dominoes of type (a, b) is even.

3. [**10**] Show that no rectangle of the form $1 \times k$ or $2 \times n$, where $4 \nmid n$, is $(1, 2)$-tileable.

4. [**35**] Show that all other rectangles of even area are $(1, 2)$-tileable.

5. [**25**] Show that for b even, there exists some M such that for every $n > M$, a $2b \times n$ rectangle is $(1, b)$-tileable.

6. [**40**] Show that for b even, there exists some M such that for every $m, n > M$ with mn even, an $m \times n$ rectangle is $(1, b)$-tileable.

7. [**25**] Prove that neither of the previous two problems holds if b is odd.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$. You may use the following facts: if $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{k-1} = 0,$$

and if $1, \omega, \ldots, \omega^{k-1}$ are distinct, then

$$(x^k - 1) = (x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{k-1}).$$

8. **[25]** Suppose x is a fifth root of unity. Find, in radical form, all possible values of

$$2x + \frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4}.$$

9. **[25]** Let $A_1A_2\ldots A_k$ be a regular k-gon inscribed in a circle of radius 1, and let P be a point lying on or inside the circumcircle. Find the maximum possible value of $(PA_1)(PA_2)\cdots(PA_k)$.

10. **[25]** Let P be a regular k-gon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of P.

11. **[25]** Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients, $a_n \neq 0$. Suppose every root of P is a root of unity, but $P(1) \neq 0$. Show that the coefficients of P are symmetric; that is, show that $a_n = a_0, a_{n-1} = a_1, \ldots$

Early Re-tile-ment [125]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that two sets S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

12. **[20]** Suppose the elements of A are either bounded below or bounded above. Show that if S tiles A, then it does so uniquely, i.e., there is a unique tiling of A by S.

13. **[35]** Let B be a set of integers either bounded below or bounded above. Then show that if S tiles all other integers $\mathbf{Z}\backslash B$, then S tiles all integers $\mathbf{Z}$.

14. **[35]** Suppose S tiles the natural numbers $\mathbf{N}$. Show that S tiles the set $\{1, 2, \ldots, k\}$ for some positive integer k.

15. **[35]** Suppose S tiles $\mathbf{N}$. Show that S is symmetric; that is, if $-S = \{-s_n, \ldots, -s_0\}$, show that $S \sim -S$.