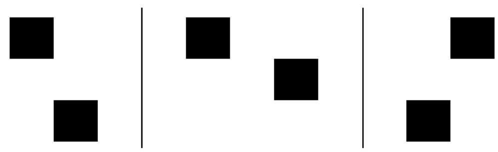
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round A — Solutions

Disconnected Domino Rally [**175**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)

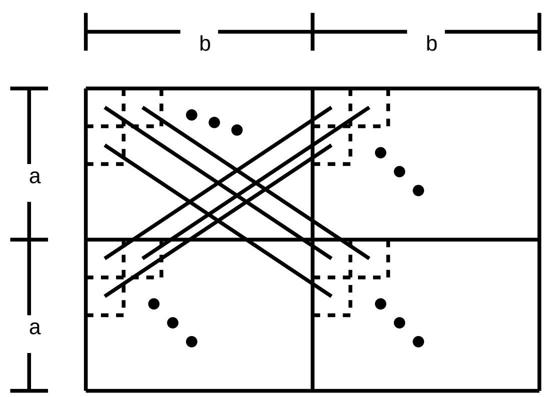


Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Prove that for any two types of dominoes, there exists a rectangle that can be tiled by dominoes of either type.

   Solution: Note that a type (a, b) domino tiles a $\max\{1, 2a\} \times 2b$ rectangle (see diagram for $a > 0$). Then both type (a, b) and type (a', b') dominoes tile a $(\max\{1, 2a\} \cdot \max\{1, 2a'\}) \times (2b \cdot 2b')$ rectangle.



2. [**25**] Suppose $0 < a \leq b$ and $4 \nmid mn$. Prove that the number of ways in which an $m \times n$ rectangle can be partitioned into dominoes of type (a, b) is even.

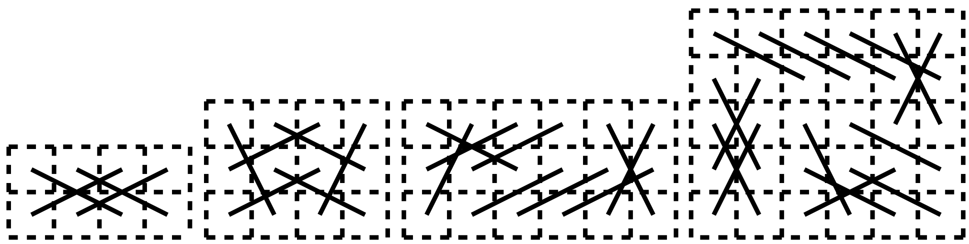
   Solution: If the rectangle is tileable, it can be partitioned into an odd number of dominoes. Consider the reflection of the partitioned rectangle over one axis. This gives another partition of the rectangle. In fact, it cannot be the same partition, for suppose it were. Then we can pair each domino with its reflected image, but since there are an odd number of dominoes, one must reflect into itself. Since $a > 0$, this is not possible. Therefore, we can pair off partitions and their reflections, and it follows that the total number of partitions is even.

3. **[10]** Show that no rectangle of the form $1 \times k$ or $2 \times n$, where $4 \nmid n$, is $(1,2)$-tileable.

Solution: The claim is obvious for $1 \times k$ rectangles. For the others, color the first two columns black, the next two white, the next two black, etc. Each $(1,2)$ domino will contain one square of each color, so in order to be tileable, the rectangle must contain the same number of black and white squares. This is the case only when $4 \mid n$.

4. **[35]** Show that all other rectangles of even area are $(1,2)$-tileable.

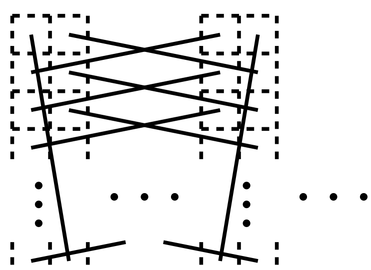
Solution: First, we demonstrate that there exist $(1,2)$-tilings of 2×4, 3×4, 3×6, and 5×6 rectangles.



Now, notice that by combining these rectangles, we can form any rectangle of even area other than those described in the previous problem: using the first rectangle, we can form any $2 \times n$ rectangle with $4 \mid n$. By combining the first two, we can form any $m \times 4$ rectangle with $m \geq 2$, and by combining the last three, we can form any $m \times 6$ rectangle with $m \geq 3$. From these, we can form any $m \times n$ rectangle with $m \geq 3$ and n even and greater than 2, completing the proof.

5. **[25]** Show that for b even, there exists some M such that for every $n > M$, a $2b \times n$ rectangle is $(1,b)$-tileable.

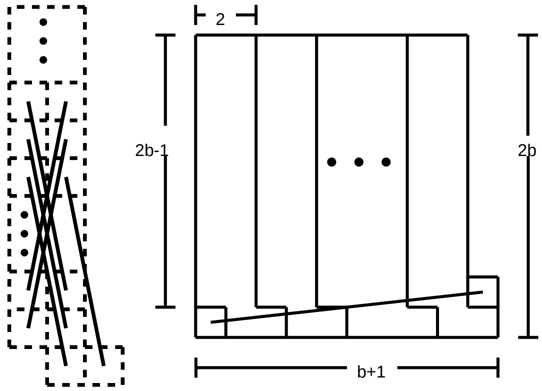
Solution: Recall from above that we can tile a $2 \times 2b$ rectangle. Four columns of a $(b+1) \times 2b$ rectangle can be tiled as shown below, and repeating this $\frac{b}{2}$ times tiles the entire rectangle. Since any integer at least b can be written as a positive linear combination of 2 and $b+1$, we can tile any $2b \times n$ rectangle for $n \geq b$.



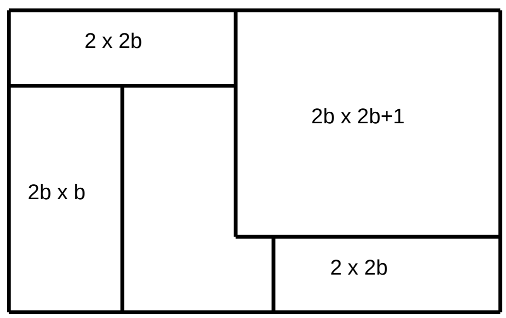
6. **[40]** Show that for b even, there exists some M such that for every $m, n > M$ with mn even, an $m \times n$ rectangle is $(1,b)$-tileable.

Solution: By the diagram below, it is possible to tile a $(2b+2) \times (4b+1)$ rectangle. Since we can already tile a $(2b+2) \times 2b$ rectangle by above, and $2b$ is relatively prime to $4b+1$, this will allow us to tile any $(2b+2) \times n$ rectangle for n sufficiently large. Combining this with the previous problem, this will allow us to tile any $m \times n$ rectangle for m and n sufficiently large and m even, completing the proof.

To tile the $(2b+2) \times (4b+1)$ rectangle, we first tile the following piece:



This is then combined with two $2 \times 2b$ rectangles, a $2b \times b$ rectangle, and a $2b \times (2b+1)$ rectangle as follows:



7. **[25]** Prove that neither of the previous two problems holds if b is odd.

   Solution: Color the grid black and white in checkerboard fashion. Then if b is odd, the two squares that make up a $(1, b)$ domino always have the same color. Therefore, for an $m \times n$ rectangle to be $(1, b)$-tileable, it must have an even number of squares of each color. Then for any M, we can choose m and n larger than M such that n is odd and $4 \nmid m$. A $2b \times n$ rectangle and an $m \times n$ rectangle then contain bn and $mn/2$ squares of each color, respectively. Since both bn and $mn/2$ are odd, neither of these rectangles is $(1, b)$-tileable.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$. You may use the following facts: if $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{k-1} = 0,$$

and if $1, \omega, \ldots, \omega^{k-1}$ are distinct, then

$$(x^k - 1) = (x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{k-1}).$$

8. **[25]** Suppose x is a fifth root of unity. Find, in radical form, all possible values of

$$2x + \frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4}.$$

Solution: Note that

$$\frac{x}{1+x^2}+\frac{x^2}{1+x^3}=\frac{x^6}{1+x^2}+\frac{x^4}{x^2+x^5}=\frac{x^4+x^6}{1+x^2}=x^4=\frac{1}{x},\text{ and}$$

$$\frac{1}{1+x}+\frac{x^3}{1+x^4}=\frac{x^5}{1+x}+\frac{x^4}{x+x^5}=\frac{x^4+x^5}{1+x}=x^4=\frac{1}{x}.$$

Therefore, the sum is just $2x+\frac{2}{x}$. If $x=1$, this is 4. Otherwise, let $y=x+\frac{1}{x}$. Then x satisfies

$$0=1+x+x^2+x^3+x^4=\left(x^2+\frac{1}{x^2}+2\right)+\left(x+\frac{1}{x}\right)-1=y^2+y-1,$$

so solving this quadratic yields $y=\frac{-1\pm\sqrt{5}}{2}$, or $2y=-1\pm\sqrt{5}$. Since each value of y can correspond to only 2 possible values of x, and there are 4 possible values of x besides 1, both of these values for y are possible, which yields the answers, 4 and $-1\pm\sqrt{5}$.

9. **[25]** Let $A_1A_2\ldots A_k$ be a regular k-gon inscribed in a circle of radius 1, and let P be a point lying on or inside the circumcircle. Find the maximum possible value of $(PA_1)(PA_2)\cdots(PA_k)$.

Solution: Place the vertices at the kth roots of unity, $1,\omega,\ldots,\omega^{k-1}$, and place P at some complex number p. Then

$$\begin{aligned}((PA_1)(PA_2)\cdots(PA_k))^2 &= \prod_{i=0}^{k-1}|p-\omega^i|^2\\ &= |p^k-1|^2,\end{aligned}$$

since $x^k-1=(x-1)(x-\omega)\cdots(x-\omega^{k-1})$. This is maximized when p^k is as far as possible from 1, which occurs when $p^k=-1$. Therefore, the maximum possible value of $(PA_1)(PA_2)\cdots(PA_k)$ is 2.

10. **[25]** Let P be a regular k-gon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of P.

Solution: Place the vertices of P at the kth roots of unity, $1,\omega,\omega^2,\ldots,\omega^{k-1}$. We will first calculate the sum of the squares of the lengths of the sides and diagonals that contain the vertex 1. This is

$$\begin{aligned}\sum_{i=0}^{k-1}|1-\omega^i|^2 &= \sum_{i=0}^{k-1}(1-\omega^i)(1-\bar{\omega}^i)\\ &= \sum_{i=0}^{k-1}(2-\omega^i-\bar{\omega}^i)\\ &= 2k-2\sum_{i=0}^{k-1}\omega^i\\ &= 2k,\end{aligned}$$

using the fact that $1+\omega+\cdots+\omega^{k-1}=0$. Now, by symmetry, this is the sum of the squares of the lengths of the sides and diagonals emanating from any vertex. Since there are k vertices and each segment has two endpoints, the total sum is $2k\cdot k/2=k^2$.

11. **[25]** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients, $a_n \neq 0$. Suppose every root of P is a root of unity, but $P(1) \neq 0$. Show that the coefficients of P are symmetric; that is, show that $a_n = a_0, a_{n-1} = a_1, \ldots$

    Solution: Since the coefficients of P are real, the complex conjugates of the roots of P are also roots of P. Now, if x is a root of unity, then $x^{-1} = \bar{x}$. But the roots of

    $$x^n P(x^{-1}) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

    are then just the complex conjugates of the roots of P, so they are the roots of P. Therefore, $P(x)$ and $x^n P(x^{-1})$ differ by a constant multiple c. Since $a_n = ca_0$ and $a_0 = ca_n$, c is either 1 or -1. But if it were -1, then

    $$P(1) = a_n + a_{n-1} + \cdots + a_0 = \frac{1}{2}((a_n + a_0) + (a_{n-1} + a_1) + \cdots + (a_0 + a_n)) = 0,$$

    a contradiction. Therefore $c = 1$, giving the result.

Early Re-tile-ment [**125**]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that two sets S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

12. **[20]** Suppose the elements of A are either bounded below or bounded above. Show that if S tiles A, then it does so uniquely, i.e., there is a unique tiling of A by S.

    Solution: Assume A is bounded below; the other case is analogous. In choosing the tiling of A, note that there is a unique choice for the set S_0 that contains the minimum element of A. But then there is a unique choice for the set S_1 that contains the minimum element of $A \backslash S_0$. Continuing in this manner, there is a unique choice for the set containing the minimum element not yet covered, so we see that the tiling is uniquely determined.

13. **[35]** Let B be a set of integers either bounded below or bounded above. Then show that if S tiles all other integers $\mathbf{Z}\backslash B$, then S tiles all integers $\mathbf{Z}$.

    Solution: Assume B is bounded above; the other case is analogous. Let a be the difference between the largest and smallest element of S. Denote the sets in the partition of $\mathbf{Z}\backslash B$ by S_k, $k \in \mathbf{Z}$, such that the minimum element of S_k, which we will denote c_k, is strictly increasing as k increases. Since B is bounded above, there exists some k_0 such that c_{k_0} is larger than all the elements of B. Let

    $$T_l = \bigcup_{k=l}^{\infty} S_k.$$

    Suppose $l \geq k_0$. Note that any element in S_k, $k < l$, is at most $c_l - 1 + a$. Therefore, T_l contains all integers that are at least $c_l + a$. Since the minimum element of T_l is c_l, T_l is completely determined by which of the integers $c_l + 1, c_l + 2, \ldots, c_l + a - 1$ it contains. This implies that there are at most 2^{a-1} possible nonequivalent sets T_l when $l \geq k_0$ (here we extend the notion of equivalence to infinite sets in the natural way.) By the Pigeonhole Principle, there must then be some $l_2 > l_1 \geq k_0$ such that $T_{l_1} \sim T_{l_2}$. But then it is easy to see that the set $S_{l_1} \cup S_{l_1+1} \cup \cdots \cup S_{l_2-1}$ tiles $\mathbf{Z}$, so S tiles $\mathbf{Z}$.

14. **[35]** Suppose S tiles the natural numbers $\mathbf{N}$. Show that S tiles the set $\{1, 2, \ldots, k\}$ for some positive integer k.

    Solution: Using the notation from above, we can find $l_1 < l_2$ such that $T_{l_1} \sim T_{l_2}$. By the same argument as in problem 12, as long as $T_{l_1} \neq \mathbf{N}$, there is a unique choice for S_{l_1-1} that contains the largest integer not in T_{l_1}. Since the same can be said for T_{l_2}, we must have that $T_{l_1-1} \sim T_{l_2-1}$. Continuing in this manner, we find that there must exist some l for which $\mathbf{N} \sim T_l$; then S tiles $\mathbf{N} \backslash T_l = \{1, 2, \cdots, c_l - 1\}$.

15. **[35]** Suppose S tiles $\mathbf{N}$. Show that S is symmetric; that is, if $-S = \{-s_n, \ldots, -s_0\}$, show that $S \sim -S$.

    Solution: Assume without loss of generality that the minimum element of S is 0. By the previous problem, S tiles the set $\{1, 2, \ldots, k\}$ for some positive integer k. Then let $P(x)$ be the polynomial $\sum_{i=0}^{n} x^{s_i}$. To say that the set $\{1, 2, \ldots, k\}$, or equivalently the set $\{0, 1, \ldots, k-1\}$, is tiled by S is to say that there is some polynomial $Q(x)$ with coefficients 0 or 1 such that $P(x)Q(x) = 1 + x + \cdots + x^{k-1} = (x^k - 1)/(x - 1)$. It follows that all the roots of $P(x)$ are roots of unity, but $P(1) \neq 0$. By question 11 above, this implies that $P(x)$ is symmetric. Therefore, $s_0 + s_n = s_1 + s_{n-1} = \cdots = s_n + s_0$, so S is symmetric.