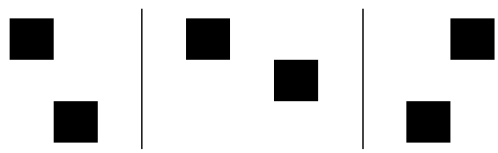
Harvard-MIT Mathematics Tournament
February 19, 2005

Team Round B

Disconnected Domino Rally [**150**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Let $0 < m \leq n$ be integers. How many different (i.e., noncongruent) dominoes can be formed by choosing two squares of an $m \times n$ array?
2. [**10**] What are the dimensions of the rectangle of smallest area that is (a, b)-tileable?
3. [**20**] Prove that every (a, b)-tileable rectangle contains a rectangle of these dimensions.
4. [**30**] Prove that an $m \times n$ rectangle is (b, b)-tileable if and only if $2b \mid m$ and $2b \mid n$.
5. [**35**] Prove that an $m \times n$ rectangle is $(0, b)$-tileable if and only if $2b \mid m$ or $2b \mid n$.
6. [**40**] Let k be an integer such that $k \mid a$ and $k \mid b$. Prove that if an $m \times n$ rectangle is (a, b)-tileable, then $2k \mid m$ or $2k \mid n$.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$.

7. [**15**] Find a real, irreducible quartic polynomial with leading coefficient 1 whose roots are all twelfth roots of unity.
8. [**25**] Let x and y be two kth roots of unity. Prove that $(x + y)^k$ is real.
9. [**30**] Let x and y be two distinct roots of unity. Prove that $x + y$ is also a root of unity if and only if $\frac{y}{x}$ is a cube root of unity.
10. [**30**] Let x, y, and z be three roots of unity. Prove that $x + y + z$ is also a root of unity if and only if $x + y = 0$, $y + z = 0$, or $z + x = 0$.

Early Re-tile-ment [**150**]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

11. [**20**] Find all sets S with minimum element 1 that tile $A = \{1, \ldots, 12\}$.

12. [**35**] Let A be a finite set with more than one element. Prove that the number of nonequivalent sets S which tile A is always even.

13. [**25**] Exhibit a set S which tiles the integers $\mathbf{Z}$ but not the natural numbers $\mathbf{N}$.

14. [**30**] Suppose that S tiles the set of all integer cubes. Prove that S has only one element.

15. [**40**] Suppose that S tiles the set of odd prime numbers. Prove that S has only one element.