IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Guts Round Solutions

1. A bear walks one mile south, one mile east, and one mile north, only to find itself where it started. Another bear, more energetic than the first, walks two miles south, two miles east, and two miles north, only to find itself where it started. However, the bears are *not* white and did *not* start at the north pole. At most how many miles apart, to the nearest .001 mile, are the two bears' starting points?

   Answer: 3.477
   Solution: Say the first bear walks a mile south, an integer $n > 0$ times around the south pole, and then a mile north. The middle leg of the first bear's journey is a circle of circumference $1/n$ around the south pole, and therefore about $\frac{1}{2n\pi}$ miles north of the south pole. (This is not exact even if we assume the Earth is perfectly spherical, but it is correct to about a micron.) Adding this to the mile that the bear walked south/north, we find that it started about $1 + \frac{1}{2n\pi}$ miles from the south pole. Similarly, the second bear started about $2 + \frac{2}{2m\pi}$ miles from the south pole for some integer $m > 0$, so they must have started at most

$$3 + \frac{1}{2n\pi} + \frac{2}{2m\pi} \leq 3 + \frac{3}{2\pi} \approx 3.477$$

   miles apart.

2. Compute the positive integer less than 1000 which has exactly 29 positive proper divisors. (Here we refer to positive integer divisors other than the number itself.)

   Answer: 720
   Solution: Recall that the number $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where the p_i are distinct primes) has exactly $(e_1+1)(e_2+1)\cdots(e_k+1)$ positive integer divisors including itself. We seek $N < 1000$ such that this expression is 30. Since $30 = 2 \cdot 3 \cdot 5$, we take $e_1 = 1, e_2 = 2, e_3 = 4$. Then we see that $N = 5^1 3^2 4^2 = 720$ is satisfactory.

3. At a nursey, 2006 babies sit in a circle. Suddenly each baby pokes the baby immediately to either its left or its right, with equal probability. What is the expected number of unpoked babies?

   Answer: $\frac{1003}{2}$
   Solution: The probability that any given baby goes unpoked is $1/4$. So the answer is $2006/4 = 1003/2$.

4. Ann and Anne are in bumper cars starting 50 meters apart. Each one approaches the other at a constant ground speed of 10 km/hr. A fly starts at Ann, flies to Anne, then back to Ann, and so on, back and forth until it gets crushed when the two bumper cars collide. When going from Ann to Anne, the fly flies at 20 km/hr; when going in the opposite direction the fly flies at 30 km/hr (thanks to a breeze). How many meters does the fly fly?

Answer: 55

Solution: Suppose that at a given instant the fly is at Ann and the two cars are $12d$ apart. Then, while each of the cars travels $4d$, the fly travels $8d$ and meets Anne. Then the fly turns around, and while each of the cars travels d, the fly travels $3d$ and meets Ann again. So, in this process described, each car travels a total of $5d$ while the fly travels $11d$. So the fly will travel $\frac{11}{5}$ times the distance traveled by each bumper car: $\frac{11}{5} \cdot \frac{50}{2} = 55$ meters.

5. Find the number of solutions in positive integers $(k; a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ to the equation

$$a_1(b_1) + a_2(b_1 + b_2) + \cdots + a_k(b_1 + b_2 + \cdots + b_k) = 7.$$

Answer: 15

Solution: Let k, $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ be a solution. Then b_1, b_1+b_2, $\ldots$, $b_1+\cdots+b_k$ is just some increasing sequence of positive integers. Considering the a_i as multiplicities, the a_i's and b_i's uniquely determine a partition of 7. Likewise, we can determine a_i's and b_i's from any partition of 7, so the number of solutions is $p(7) = 15$.

6. Suppose ABC is a triangle such that $AB = 13, BC = 15$, and $CA = 14$. Say D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{DF}$. Compute the area of triangle EFG.

Answer: $\frac{21}{4}$

Solution: By Heron's formula, $[ABC] = \sqrt{21(21-15)(21-14)(21-13)} = 84$. Now, unwinding the midpoint conditions yields $[EFG] = \frac{[DEF]}{2} = \frac{[BDE]}{4} = \frac{[ABD]}{8} = \frac{[ABC]}{16} = \frac{84}{16} = \frac{21}{4}$.

7. Find all real numbers x such that

$$x^2 + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = 10.$$

Answer: $-\sqrt{14}$

Solution: Evidently x^2 must be an integer. Well, there aren't that many things to check, are there? Among positive x, $\sqrt{8}$ is too small and $\sqrt{9}$ is too big; among negative x, $-\sqrt{15}$ is too small and $-\sqrt{13}$ is too big.

8. How many ways are there to label the faces of a regular octahedron with the integers 1–8, using each exactly once, so that any two faces that share an edge have numbers that are relatively prime? Physically realizable rotations are considered indistinguishable, but physically unrealizable reflections are considered different.

Answer: 12

Solution: Well, instead of labeling the faces of a regular octahedron, we may label the vertices of a cube. Then, as no two even numbers may be adjacent, the even numbers better form a regular tetrahedron, which can be done in 2 ways (because rotations are indistiguishable but reflections are different). Then 3 must be opposite 6, and the remaining numbers — 1, 5, 7 — may be filled in at will, in $3! = 6$ ways. The answer is thus $2 \times 6 = 12$.

9. Four unit circles are centered at the vertices of a unit square, one circle at each vertex. What is the area of the region common to all four circles?

Answer: $\frac{\pi}{3} + 1 - \sqrt{3}$
Solution: The desired region consists of a small square and four "circle segments," i.e. regions of a circle bounded by a chord and an arc. The side of this small square is just the chord of a unit circle that cuts off an angle of 30°, and the circle segments are bounded by that chord and the circle. Using the law of cosines (in an isosceles triangle with unit leg length and vertex angle 30°), we find that the square of the length of the chord is equal to $2 - \sqrt{3}$. We can also compute the area of each circle segment, namely $\frac{\pi}{12} - \frac{1}{2}(1)(1)\sin 30^\circ = \frac{\pi}{12} - \frac{1}{4}$. Hence, the desired region has area $2 - \sqrt{3} + 4\left(\frac{\pi}{12} - \frac{1}{4}\right) = \frac{\pi}{3} + 1 - \sqrt{3}$.

10. Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

Answer: 9
Solution: We see the size of the set $f^{-1}(f^{-1}(f^{-1}(f^{-1}(3))))$. Note that $f(x) = (x-1)^2 - 1 = 3$ has two solutions: $x = 3$ and $x = -1$, and that the fixed points $f(x) = x$ are $x = 3$ and $x = 0$. Therefore, the number of real solutions is equal to the number of distinct real numbers c such that $c = 3$, $c = -1$, $f(c) = -1$ or $f(f(c)) = -1$, or $f(f(f(c))) = -1$. The equation $f(x) = -1$ has exactly one root $x = 1$. Thus, the last three equations are equivalent to $c = 1, f(c) = 1$, and $f(f(c)) = 1$. $f(c) = 1$ has two solutions, $c = 1 \pm \sqrt{2}$, and for each of these two values c there are two preimages. It follows that the answer is $1 + 1 + 1 + 2 + 4 = 9$.

11. Find all positive integers $n > 1$ for which $\frac{n^2+7n+136}{n-1}$ is the square of a positive integer.

Answer: 5, 37
Solution: Write $\frac{n^2+7n+136}{n-1} = n + \frac{8n+136}{n-1} = n + 8 + \frac{144}{n-1} = 9 + (n-1) + \frac{144}{(n-1)}$. We seek to find p and q such that $pq = 144$ and $p + q + 9 = k^2$. The possibilities are seen to be $1 + 144 + 9 = 154$, $2 + 72 + 9 = 83$, $3 + 48 + 9 = 60$, $4 + 36 + 9 = 49$, $6 + 24 + 9 = 39$, $8 + 18 + 9 = 35$, $9 + 16 + 9 = 34$, and $12 + 12 + 9 = 33$. Of these, $\{p, q\} = \{4, 36\}$ is the only solution to both equations. Hence $n - 1 = 4, 36$ and $n = 5, 37$.

12. For each positive integer n let S_n denote the set $\{1, 2, 3, \ldots, n\}$. Compute the number of triples of subsets A, B, C of S_{2006} (not necessarily nonempty or proper) such that A is a subset of B and $S_{2006} - A$ is a subset of C.

Answer: 2^{4012}
Solution: Let A_o, B_o, C_o be sets satisfying the said conditions. Note that $1 \in A_o$ implies that $1 \in B_o$ and $1 \notin S_{2006} - A_o$ so that 1 may or may not be in C_o. Also,

$1 \notin A_o$ implies that $1 \in S_{2006} - A_o \subset C_o$ while 1 may or may not be in B_o. Thus there are four possibilities for the distribution of 1, and since the same argument holds independently for 2, 3, ..., 2006, the answer is 4^{2006} or 2^{4012}.

13. Let Z be as in problem 15. Let X be the greatest integer such that $|XZ| \leq 5$. Find X.

    Answer: 2
    Solution: Problems 13–15 go together. See below.

14. Let X be as in problem 13. Let Y be the number of ways to order X crimson flowers, X scarlet flowers, and X vermillion flowers in a row so that no two flowers of the same hue are adjacent. (Flowers of the same hue are mutually indistinguishable.) Find Y.

    Answer: 30
    Solution: Problems 13–15 go together. See below.

15. Let Y be as in problem 14. Find the maximum Z such that three circles of radius $\sqrt{Z}$ can simultaneously fit inside an equilateral triangle of area Y without overlapping each other.

    Answer: $10\sqrt{3} - 15$
    Solution: We first find that, in problem 15, each of the circles of radius $\sqrt{Z}$ is the incircle of a 30°-60°-90° triangle formed by cutting the equilateral one in half. The equilateral triangle itself has sidelength $\frac{2\sqrt{Y}}{\sqrt[4]{3}}$, so the said inradius is

$$\sqrt{Z} = \frac{1+\sqrt{3}-2}{2} \cdot \frac{1}{2} \cdot \frac{2\sqrt{Y}}{\sqrt[4]{3}},$$

so that

$$Z = \frac{(-1+\sqrt{3})^2}{4\sqrt{3}} Y = \frac{4-2\sqrt{3}}{4\sqrt{3}} Y = \frac{2\sqrt{3}-3}{6} Y.$$

Now we guess that $X = 2$ and see that, miraculously, everything works: in the problem 14, say a crimson flower is placed first. Then there are 2 possibilities for $C\_C\_\_\_$, 4 for $C\_\_C\_\_$, 2 for $C\_\_\_C\_$, and 2 for $C\_\_\_\_C$, giving a total of 10. Of course, the first flower can be of any of the three hues, so $Y = 3 \cdot 10 = 30$. We compute Z and check X in a straightforward manner.

If $X > 2$, then $Y > 30$, and $Z > 10\sqrt{3} - 15$, with the result that $X \leq 2$, a contradiction. Assuming $X < 2$ results in a similar contradiction.

16. A sequence $a_1, a_2, a_3, \ldots$ of positive reals satisfies $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine all a_1 such that $a_i = \frac{\sqrt{6}+\sqrt{2}}{4}$ for some positive integer i.

    Answer: $\dfrac{\sqrt{2}+\sqrt{6}}{2}, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}$
    Solution: Clearly $a_1 < 1$, or else $1 \leq a_1 \leq a_2 \leq a_3 \leq \ldots$. We can therefore write $a_1 = \cos\theta$ for some $0 < \theta < 90°$. Note that $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$, and $\cos 15° =$

$\frac{\sqrt{6}+\sqrt{2}}{4}$. Hence, the possibilities for a_1 are $\cos 15°$, $\cos 30°$, and $\cos 60°$, which are $\frac{\sqrt{2}+\sqrt{6}}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{1}{2}$.

17. Begining at a vertex, an ant is crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities equally likely.) What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

Answer: $\frac{2^{2005}+1}{3\cdot 2^{2006}}$
Solution: For each nonnegative integer n, let a_n, b_n, and c_n denote the respective probabilities that the ant is where it began, at a neighbor of where it began, or is opposite where it began after moving along n edges. We seek a_{2006}. We have $a_0 = 1$ and $b_0 = c_0 = 0$. We also have the recursive system

$$\begin{aligned} a_n &= \frac{b_{n-1}}{4} \\ b_n &= a_{n-1} + \frac{b_{n-1}}{2} + c_{n-1} \\ c_n &= \frac{b_{n-1}}{4} \end{aligned}$$

for integers $n \geq 1$. Substituting into the second equation we have $b_n = \frac{b_{n-1}}{2} + \frac{b_{n-2}}{2}$ for $n \geq 2$. Solving the characteristic equation $x^2 - \frac{x}{2} - \frac{1}{2} = 0$ for $x = 1, \frac{-1}{2}$, we write $b_n = a \cdot 1^n + b(-1/2)^n$. Using $b_0 = 0, b_1 = 1$, we compute

$$b_n = \frac{2}{3} \cdot (1 - (-1/2)^n)$$

From which we find $a_{2006} = \frac{b_{2005}}{4} = \frac{1}{6}\left(1 + \frac{1}{2^{2005}}\right) = \frac{2^{2005}+1}{3\cdot 2^{2006}}$.

18. Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Points P and Q are the midpoints of $\overline{BC}$ and $\overline{DA}$. Compute PQ^2.

Answer: $\frac{116}{35}$
Solution: Construct $\overline{AC}, \overline{AQ}, \overline{BQ}, \overline{BD}$, and let R denote the intersection of $\overline{AC}$ and $\overline{BD}$. Because $ABCD$ is cyclic, we have that $\triangle ABR \sim \triangle DCR$ and $\triangle ADR \sim \triangle BCR$. Thus, we may write $AR = 4x, BR = 2x, CR = 6x, DR = 12x$. Now, Ptolemy applied to $ABCD$ yields $140x^2 = 1 \cdot 3 + 2 \cdot 4 = 11$. Now $\overline{BQ}$ is a median in triangle ABD. Hence, $BQ^2 = \frac{2BA^2+2BD^2-AD^2}{4}$. Likewise, $CQ^2 = \frac{2CA^2+2CD^2-DA^2}{4}$. But PQ is a median in triangle BQC, so $PQ^2 = \frac{2BQ^2+2CQ^2-BC^2}{4} = \frac{AB^2+BD^2+CD^2+CA^2-BC^2-AD^2}{4} = \frac{(196+100)x^2+1^2+3^2-2^2-4^2}{4} = \frac{148x^2-5}{2} = \frac{148\cdot\frac{11}{140}-5}{2} = \frac{116}{35}$.

Another solution is possible. Extend $\overline{AD}$ and $\overline{BC}$ past A and B to their intersection S. Use similar triangles SAB and SCD, and similar triangles SAC and SBD to compute SA and SB, then apply the Law of Cosines twice, first to compute the cosine of $\angle A$ and then to compute PQ^2.

19. Let ABC be a triangle with $AB = 2, CA = 3, BC = 4$. Let D be the point diametrically opposite A on the circumcircle of ABC, and let E lie on line AD such that D is the midpoint of $\overline{AE}$. Line l passes through E perpendicular to $\overline{AE}$, and F and G are the intersections of the extensions of $\overline{AB}$ and $\overline{AC}$ with l. Compute FG.

Answer: $\frac{1024}{45}$
Solution: Using Heron's formula we arrive at $[ABC] = \frac{3\sqrt{15}}{4}$. Now invoking the relation $[ABC] = \frac{abc}{4R}$ where R is the circumradius of ABC, we compute $R^2 = \left(\frac{2\cdot 3}{[ABC]^2}\right) = \frac{64}{15}$. Now observe that $\angle ABD$ is right, so that $BDEF$ is a cyclic quadrilateral. Hence $AB \cdot AF = AD \cdot AE = 2R \cdot 4R = \frac{512}{15}$. Similarly, $AC \cdot AG = \frac{512}{15}$. It follows that $BCGF$ is a cyclic quadrilateral, so that triangles ABC and AGF are similar. Then $FG = BC \cdot \frac{AF}{AC} = 4 \cdot \frac{512}{2\cdot 15\cdot 3} = \frac{1024}{45}$

20. Compute the number of real solutions (x, y, z, w) to the system of equations:

$$x = z + w + zwx \qquad z = x + y + xyz$$
$$y = w + x + wxy \qquad w = y + z + yzw$$

Answer: 5
Solution: The first equation rewrites as $x = \frac{w+z}{1-wz}$, which is a fairly strong reason to consider trigonometric substitution. Let $x = \tan(a), y = \tan(b), z = \tan(c)$, and $w = \tan(d)$, where $-90° < a, b, c, d < 90°$. Under modulo $180°$, we find $a \equiv c + d$; $b \equiv d + a$; $c \equiv a + b$; $d \equiv b + c$. Adding all of these together yields $a + b + c + d \equiv 0$. Then $a \equiv c + d \equiv -a - b$ so $b \equiv -2a$. Similarly, $c \equiv -2b$; $d \equiv -2c$; $d \equiv -2a$. Hence, $c \equiv -2b \equiv 4a$, $d \equiv -2c \equiv -8a$, and $a \equiv -2d \equiv 16a$, so the only possible solutions are $(a, b, c, d) \equiv (t, -2t, 4t, -8t)$ where $15t \equiv 0$. Checking, these, we see that actually $5t \equiv 0$, which yields 5 solutions. Our division by $1 - yz$ is valid since $1 - yz = 0$ iff $yz = 1$, but $x = y + z + xyz$ so $y = -z$, which implies that $yz \leq 0 < 1$, which is impossible. (The solutions we have computed are in fact (0,0,0,0) and the cyclic permutations of $(\tan(36°), \tan(-72°), \tan(-36°), \tan(72°))$.)

21. Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

Answer: 84
Solution: Let $Q(z)$ denote the polynomial divisor. We need that the roots of Q are k-th roots of unity. With this in mind, we might observe that solutions to $z^7 = 1$ and $z \neq 1$ are roots of Q, which leads to its factorization. Alternatively, we note that

$$(z - 1)Q(z) = z^{11} - z^9 + z^7 - z^4 + z^2 - 1 = (z^4 - z^2 + 1)(z^7 - 1)$$

Solving for the roots of the first factor, $z^2 = \frac{1+i\sqrt{3}}{2} = \pm\text{cis}\pi/3$ (we use the notation $\text{cis}(x) = \cos(x) + i\sin(x)$) so that $z = \pm\text{cis}(\pm\pi/6)$. These are primitive 12-th roots of unity. The other roots of $Q(z)$ are the primitive 7-th roots of unity (we introduced $z = 1$ by multiplication.) It follows that the answer is $\text{lcm}[12, 7] = 84$.

22. Let $f(x)$ be a degree 2006 polynomial with complex roots $c_1, c_2, \ldots, c_{2006}$, such that the set

$$\{|c_1|, |c_2|, \ldots, |c_{2006}|\}$$

consists of exactly 1006 distinct values. What is the minimum number of real roots of $f(x)$?

Answer: 6
Solution: The complex roots of the polynomial must come in pairs, c_i and $\overline{c_i}$, both of which have the same absolute value. If n is the number of distinct absolute values $|c_i|$ corresponding to those of non-real roots, then there are at least $2n$ non-real roots of $f(x)$. Thus $f(x)$ can have at most $2006 - 2n$ real roots. However, it must have at least $1006 - n$ real roots, as $|c_i|$ takes on $1006 - n$ more values. By definition of n, these all correspond to real roots. Therefore $1006 - n \leq \#$ real roots $\leq 2006 - 2n$, so $n \leq 1000$, and $\#$ real roots $\geq 1006 - n \geq 6$. It is easy to see that equality is attainable.

23. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by $a_0 = 21, a_1 = 35$, and $a_{n+2} = 4a_{n+1} - 4a_n + n^2$ for $n \geq 2$. Compute the remainder obtained when a_{2006} is divided by 100.

Answer: 0
Solution: No pattern is evident in the first few terms, so we look for a formula for a_n. If we write $a_n = An^2 + Bn + C + b_n$ and put $b_{n+2} = 4b_{n+1} - 4b_n$. Rewriting the original recurrence, we find

$$\begin{aligned} &An^2 + (4A + B)n + (4A + 2B + C) + b_{n+2} \\ &\quad = 4\left(An^2 + (2A + B)n + (A + B + C) + b_{n+1}\right) - 4\left(An^2 + Bn + C + b_n\right) + n^2 \\ &\quad = n^2 + 8An + (4A + 4B) + 4b_{n+1} - 4b_n \end{aligned}$$

Solving, $A = 1, B = 4, C = 8$. With this information, we can solve for $b_0 = 1$ and $b_1 = 6$. Since the characteristic equation of the recurrence of the b_i is $x^2 - 4x + 4 = (x - 2)^2 = 0$, we have $b_n = (Dn + E) \cdot 2^n$ for some constants D and E. Using the known values b_0 and b_1, we compute $D = 2$ and $E = 1$, and finally

$$a_n = n^2 + 4n + 8 + (2n + 1) \cdot 2^n$$

Now, taking modulo 100, we have $a_{2006} \equiv 6^2 + 4 \cdot 6 + 8 + 13 \cdot 2^{2006} \pmod{100}$. Evidently $2^{2006} \equiv 0 \pmod 4$, but by Euler's theorem $2^{\phi(25)} \equiv 2^{20} \equiv 1 \pmod{25}$, and so $2^{2006} \equiv 2^6 \equiv 14 \pmod{25}$. Now the Chinese remainder theorem yields $2^{2006} \equiv 64 \pmod{100}$, and we compute $a_{2006} \equiv 36 + 24 + 8 + 13 \cdot 64 \equiv 0 \pmod{100}$.

24. Two 18-24-30 triangles in the plane share the same circumcircle as well as the same incircle. What's the area of the region common to both the triangles?

Answer: 132
Solution: Notice, first of all, that 18-24-30 is 6 times 3-4-5, so the triangles are right. Thus, the midpoint of the hypotenuse of each is the center of their common circumcircle, and the inradius is $\frac{1}{2}(18 + 24 - 30) = 6$. Let one of the triangles be ABC,

where $\angle A < \angle B < \angle C = 90°$. Now the line ℓ joining the midpoints of sides AB and AC is tangent to the incircle, because it is the right distance (12) from line BC. So, the hypotenuse of the other triangle lies along ℓ. We may formulate this thus: *The hypotenuse of each triangle is parallel to the shorter leg, and therefore perpendicular to the longer leg, of the other.* Now it is not hard to see, as a result of these parallel- and perpendicularisms, that the other triangle "cuts off" at each vertex of $\triangle ABC$ a smaller, similar right triangle. If we compute the dimensions of these smaller triangles, we find that they are as follows: 9-12-15 at A, 6-8-10 at B, and 3-4-5 at C. The total area chopped off of $\triangle ABC$ is thus

$$\frac{9 \cdot 12}{2} + \frac{6 \cdot 8}{2} + \frac{3 \cdot 4}{2} = 54 + 24 + 6 = 84.$$

The area of $\triangle ABC$ is $18 \cdot 24/2 = 216$. The area of the region common to both the original triangles is thus $216 - 84 = 132$.

25. Points A, C, and B lie on a line in that order such that $AC = 4$ and $BC = 2$. Circles ω_1, ω_2, and ω_3 have $\overline{BC}, \overline{AC}$, and $\overline{AB}$ as diameters. Circle Γ is externally tangent to ω_1 and ω_2 at D and E respectively, and is internally tangent to ω_3. Compute the circumradius of triangle CDE.

Answer: $\frac{2}{3}$
Solution: Let the center of ω_i be O_i for $i = 1, 2, 3$ and let O denote the center of Γ. Then O, D, and O_1 are collinear, as are O, E, and O_2. Denote by F the point of tangency between Γ and ω_3; then F, O, and O_3 are collinear. Writing r for the radius of Γ we have $OO_1 = r + 2, OO_2 = r + 1, OO_3 = 3 - r$. Now since $O_1O_3 = 1$ and $O_3O_2 = 2$, we apply Stewart's theorem:

$$\begin{aligned} OO_1^2 \cdot O_2O_3 + OO_2^2 \cdot O_1O_3 &= OO_3^2 \cdot O_1O_2 + O_1O_3 \cdot O_3O_2 \cdot O_1O_2 \\ 2(r+2)^2 + (r+1)^2 &= 3(3-r)^2 + 1 \cdot 2 \cdot 3 \end{aligned}$$

We find $r = \frac{6}{7}$. Now the key observation is that the circumcircle of triangle CDE is the incircle of triangle OO_1O_2. We easily compute the sides of OO_1O_2 to be $\frac{13}{7}, \frac{20}{7}$, and 3. By Heron's formula, the area of OO_1O_2 is $\frac{18}{7}$, but the semiperimeter is $\frac{27}{7}$, so the desired radius is $\frac{2}{3}$.

26. Let $a \geq b \geq c$ be real numbers such that

$$\begin{aligned} a^2bc + ab^2c + abc^2 + 8 &= a + b + c \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 &= 2 + ab + bc + ca \end{aligned}$$

If $a + b + c > 0$, then compute the integer nearest to a^5.

Answer: 1279
Solution: We factor the first and third givens, obtaining the system

$$\begin{aligned} a^2bc + ab^2c + abc^2 - a - b - c = (abc - 1)(a + b + c) &= -8 \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc = (ab + bc + ca)(a + b + c) &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 - ab - bc - ca = (abc - 1)(ab + bc + ca) &= 2 \end{aligned}$$

Writing $X = a+b+c, Y = ab+bc+ca, Z = abc-1$, we have $XZ = -8, XY = -4, YZ = 2$. Multiplying the three yields $(XYZ)^2 = 64$ from which $XYZ = \pm 8$. Since we are given $X > 0$, multiplying the last equation by X we have $2X = XYZ = \pm 8$. Evidently $XYZ = 8$ from which $X = 4, Y = -1, Z = -2$. We conclude that a, b, c are the roots of the polynomial $P(t) = t^3 - 4t^2 - t + 1$. Thus, $P(a) = a^3 - 4a^2 - a + 1 = 0$, and also $P(b) = P(c) = 0$. Now since $P(1/2) = -\frac{3}{8}$, $P(0) = 1$ and $P(-2/3) = -\frac{11}{27}$, we conclude that $-2/3 < c < 0 < b < 1/2 < a$. It follows that $|b^5 + c^5| < \frac{1}{2}$. Thus, we compute $a^5 + b^5 + c^5$.

Defining $S_n = a^n + b^n + c^n$, we have $S_{n+3} = 4S_{n+2} + S_{n+1} - S_n$ for $n \geq 0$. Evidently $S_0 = 3, S_1 = 4, S_2 = (a+b+c)^2 - 2(ab+bc+ca) = 18$. Then $S_3 = 4 \cdot 18 + 4 - 3 = 73$, $S_4 = 4 \cdot 73 + 18 - 4 = 306$, and $S_5 = 4 \cdot 306 + 73 - 18 = 1279$. Since $|b^5 + c^5| < \frac{1}{2}$, we conclude that $|S_5 - a^5| < \frac{1}{2}$, or that 1279 is the integer nearest to a^5.

27. Let N denote the number of subsets of $\{1, 2, 3, \ldots, 100\}$ that contain more prime numbers than multiples of 4. Compute the largest integer k such that 2^k divides N.

Answer: 52

Solution: Let S denote a subset with the said property. Note that there are 25 multiples of 4 and 25 primes in the set $\{1, 2, 3, \ldots, 100\}$, with no overlap between the two. Let T denote the subset of 50 numbers that are neither prime nor a multiple of 4, and let U denote the 50 other numbers. Elements of T can be arbitrarily included in or excluded by S. Consider $S \cap U = S_1$ and $U - S = S_2$ (the set difference is defined to be all elements of U that are not in S.) S_1 and S_2 are two disjoint sets such that $U = S_1 \cap S_2$. If S_1 contains more multiples of 4 than primes, then S_2 contains more primes than multiples of 4, and conversely. Furthermore, S_1 contains an equal number of primes and multiples of 4 if and only if S_2 contains equal numbers as well. Let V denote an arbitrary subset of T. It follows from examining pairs of sets $V \cup S_1$ and $V \cup S_2$ that

$$\begin{aligned} N &= 2^{50} \cdot \frac{1}{2}\left(2^{50} - \sum_{k=0}^{25} \binom{25}{k}^2\right) \\ &= 2^{49} \cdot \left(2^{50} - \binom{50}{25}\right) \end{aligned}$$

Since $50!$ is divisible by 2 exactly $25 + 12 + 6 + 3 + 1 = 47$ times while $25!$ is divisible by 2 exactly $12 + 6 + 3 + 1 = 22$ times, it follows that $\binom{50}{25}$ is divisible by 2 exactly 3 times, so the answer is $49 + 3 = 52$.

28. A pebble is shaped as the intersection of a cube of side length 1 with the solid sphere tangent to all of the cube's edges. What is the surface area of this pebble?

Answer: $\dfrac{6\sqrt{2} - 5}{2}\pi$

Solution: Imagine drawing the sphere and the cube. Take a cross section, with a plane parallel to two of the cube's faces, passing through the sphere's center. In this cross section, the sphere looks like a circle, and the cube looks like a square (of side length 1) inscribed in that circle. We can now calculate that the sphere has diameter

$d := \sqrt{2}$ and surface area $S := \pi d^2 = 2\pi$, and that the sphere protrudes a distance of $x := \frac{\sqrt{2}-1}{2}$ out from any given face of the cube.

It is known that the surface area chopped off from a sphere by any plane is proportional to the perpendicular distance thus chopped off. Thus, each face of the cube chops of a fraction $\frac{x}{d}$ of the sphere's surface. The surface area of the pebble contributed by the sphere is thus $S \cdot (1 - 6 \cdot \frac{x}{d})$, whereas the cube contributes 6 circles of radius $\frac{1}{2}$, with total area $6 \cdot \pi \left(\frac{1}{2}\right)^2 = \frac{3}{2}\pi$. The pebble's surface area is therefore

$$S \cdot \left(1 - 6 \cdot \frac{x}{d}\right) + \frac{3}{2}\pi = 2\pi \cdot \left(1 - 6 \cdot \frac{\sqrt{2}-1}{2\sqrt{2}}\right) + \frac{3}{2}\pi = \frac{6\sqrt{2}-5}{2}\pi.$$

29. Find the area in the first quadrant bounded by the hyperbola $x^2 - y^2 = 1$, the x-axis, and the line $3x = 4y$.

Answer: $\frac{\ln 7}{4}$
Solution: Convert to polar coordinates: the hyperbola becomes

$$1 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos(2\theta),$$

so, letting $\alpha := \arctan(3/4)$, the area is

$$S := \int_0^\alpha \frac{r^2}{2} d\theta = \frac{1}{2}\int_0^\alpha \sec(2\theta) d\theta = \frac{1}{4}\ln|\sec(2\theta) + \tan(2\theta)|\Big|_0^\alpha.$$

Now

$$\tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{3/2}{7/16} = \frac{24}{7},$$
$$\sec(2\alpha) = \sqrt{1 + \tan^2(2\alpha)} = \frac{25}{7},$$

so

$$S = \frac{1}{4}\left(\ln\left|\frac{25}{7} + \frac{24}{7}\right| - \ln|1+0|\right) = \frac{\ln 7}{4}.$$

30. ABC is an acute triangle with incircle ω. ω is tangent to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. P is a point on the altitude from A such that Γ, the circle with diameter $\overline{AP}$, is tangent to ω. Γ intersects $\overline{AC}$ and $\overline{AB}$ at X and Y respectively. Given $XY = 8$, $AE = 15$, and that the radius of Γ is 5, compute $BD \cdot DC$.

Answer: $\frac{675}{4}$
Solution: By the Law of Sines we have $\sin\angle A = \frac{XY}{AP} = \frac{4}{5}$. Let I, T, and Q denote the center of ω, the point of tangency between ω and Γ, and the center of Γ respectively. Since we are told ABC is acute, we can compute $\tan\angle\frac{A}{2} = \frac{1}{2}$. Since $\angle EAI = \frac{A}{2}$ and $\overline{AE}$ is tangent to ω, we find $r = \frac{AE}{2} = \frac{15}{2}$. Let H be the foot of the altitude from A to $\overline{BC}$. Define h_T to be the homothety about T which sends Γ to ω. We have $h_T(\overline{AQ}) = \overline{DI}$, and conclude that A, T, and D are collinear. Now since $\overline{AP}$ is a diameter of Γ, $\angle PAT$ is right, implying that $DTHP$ is cyclic. Invoking Power of

a Point twice, we have $225 = AE^2 = AT \cdot AD = AP \cdot AH$. Because we are given radius of Γ we can find $AP = 10$ and $AH = \frac{45}{2} = h_a$. If we write a, b, c, s in the usual manner with respect to triangle ABC, we seek $BD \cdot DC = (s-b)(s-c)$. But recall that Heron's formula gives us

$$\sqrt{s(s-a)(s-b)(s-c)} = K$$

where K is the area of triangle ABC. Writing $K = rs$, we have $(s-b)(s-c) = \frac{r^2 s}{s-a}$. Knowing $r = \frac{15}{2}$, we need only compute the ratio $\frac{s}{a}$. By writing $K = \frac{1}{2} a h_a = rs$, we find $\frac{s}{a} = \frac{h_a}{2r} = \frac{3}{2}$. Now we compute our answer, $\frac{r^2 s}{s-a} = \left(\frac{15}{2}\right)^2 \cdot \frac{\frac{s}{a}}{\frac{s}{a} - 1} = \frac{675}{4}$.

31. Let A be as in problem 33. Let W be the sum of all positive integers that divide A. Find W.

Answer: 8
Solution: Problems 31–33 go together. See below.

32. In the alphametic $WE \times EYE = SCENE$, each different letter stands for a different digit, and no word begins with a 0. The W in this problem has the same value as the W in problem 31. Find S.

Answer: 5
Solution: Problems 31–33 go together. See below.

33. Let W, S be as in problem 32. Let A be the least positive integer such that an acute triangle with side lengths S, A, and W exists. Find A.

Answer: 7
Solution: There are two solutions to the alphametic in problem 32: $36 \times 686 = 24696$ and $86 \times 636 = 54696$. So (W, S) may be $(3, 2)$ or $(8, 5)$. If $(W, S) = (3, 2)$, then by problem (3) $A = 3$, but then by problem 31 $W = 4$, a contradiction. So, (W, S) must be $(8, 5)$. By problem 33, $A = 7$, and this indeed checks in problem 31.

34. In bridge, a standard 52-card deck is dealt in the usual way to 4 players. By convention, each hand is assigned a number of "points" based on the formula

$$4 \times (\# \text{ A's}) + 3 \times (\# \text{ K's}) + 2 \times (\# \text{ Q's}) + 1 \times (\# \text{ J's}).$$

Given that a particular hand has exactly 4 cards that are A, K, Q, or J, find the probability that its point value is 13 or higher.

Answer: $\frac{197}{1820}$
Solution: Obviously, we can ignore the cards lower than J. Simply enumerate the ways to get at least 13 points: AAAA (1), AAAK (16), AAAQ (16), AAAJ (16), AAKK (36), AAKQ (96), AKKK (16). The numbers in parentheses represent the number of ways to choose the suits, given the choices for the values. We see that there are a total of $1 + 16 + 16 + 16 + 36 + 96 + 16 = 197$ ways to get at least 13. There are a total of $\binom{16}{4} = 1820$ possible ways to choose 4 cards from the 16 total A's, K's, Q's, and J's. Hence the answer is 197/1820.

35. A sequence is defined by $A_0 = 0, A_1 = 1, A_2 = 2$, and, for integers $n \geq 3$,

$$A_n = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

Compute $\lim_{N\to\infty} A_N$.

Answer: $\frac{13}{6} - \frac{\pi^2}{12}$.
Solution: If we sum the given equation for $n = 3, 4, 5, \ldots, N$, we obtain

$$\sum_{n=3}^{N} A_n = \sum_{n=3}^{N} \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

This reduces dramatically to

$$A_N + \frac{2A_{N-1}}{3} + \frac{A_{N-2}}{3} = A_2 + \frac{2A_1}{3} + \frac{A_0}{3} + \sum_{n=3}^{N} \frac{1}{n^4 - n^2} \qquad (*)$$

Let $\lim_{N\to\infty} A_N = L$. Under this limit, the left hand side of (*) is simply $2L$. We compute the sum on the right with the help of partial fractions

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=3}^{N} \frac{1}{n^4 - n^2} &= \sum_{n=3}^{\infty} \frac{1}{n^2 - 1} - \frac{1}{n^2} \\
&= \left(\sum_{n=3}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right) + \frac{1}{1^2} + \frac{1}{2^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) + \frac{5}{4} - \frac{\pi^2}{6} \\
&= \frac{5}{3} - \frac{\pi^2}{6}
\end{aligned}$$

With this we easily find $L = \frac{1}{2} \cdot \left(2 + \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{5}{3} - \frac{\pi^2}{6}\right) = \frac{13}{6} - \frac{\pi^2}{12}$, and we are done.

36. Four points are independently chosen uniformly at random from the interior of a regular dodecahedron. What is the probability that they form a tetrahedron whose interior contains the dodecahedron's center?

Answer: $\frac{1}{8}$
Solution: Situate the origin O at the dodecahedron's center, and call the four random points P_i, where $1 \leq i \leq 4$.

To any tetrahedron $P_1P_2P_3P_4$ we can associate a quadruple $(\epsilon_{(ijk)})$, where (ijk) ranges over all conjugates of the cycle (123) in the alternating group A_4: ϵ_{ijk} is the sign of the directed volume $[OP_iP_jP_k]$. Assume that, for a given tetrahedron $P_1P_2P_3P_4$, all members of its quadruple are nonzero (this happens with probability 1). For $1 \leq i \leq 4$, if we replace P_i with its reflection through the origin, the three members of the tetrahedron's quadruple that involve P_i all flip sign, because each $[OP_iP_jP_k]$ is a linear function of the vector $\overrightarrow{OP_i}$. Thus, if we consider the 16 sister tetrahedra obtained by choosing independently whether to flip each P_i through the origin, the quadruples range through all 16 possibilities (namely, all the quadruples consisting of ± 1s). Two of these 16 tetrahedra, namely those with quadruples $(1,1,1,1)$ and $(-1,-1,-1,-1)$, will contain the origin.

So the answer is $2/16 = 1/8$.

37. Compute

$$\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3+7n^2+14n+8)}$$

Answer: $\frac{137}{24} - 8\ln 2$
Solution: First, we manipulate using partial fractions and telescoping:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3+7n^2+14n+8)} &= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{2}{n+1} - \frac{1}{n+2} - \frac{1}{n+4}\right) \\ &= \frac{1}{4} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^n \cdot (n+4)}\end{aligned}$$

Now, consider the function $f(r,k) := \sum_{n=1}^{\infty} \frac{r^n}{n^k}$. We have

$$\begin{aligned}\frac{\partial f(r,k)}{\partial r} &= \frac{\partial}{\partial r}\sum_{n=1}^{\infty}\frac{r^n}{n^k} = \sum_{n=1}^{\infty}\frac{\partial}{\partial r}\left[\frac{r^n}{n^k}\right] = \sum_{n=1}^{\infty}\frac{r^{n-1}}{n^{k-1}} = \frac{1}{r}f(r,k-1) \\ \frac{df(r,1)}{dr} &= \frac{1}{r}\sum_{n=1}^{\infty}\frac{r^n}{n^0} = \frac{1}{r}\cdot\frac{r}{1-r} = \frac{1}{1-r} \\ f(r,1) &= \int \frac{dr}{1-r} = -\ln(1-r) + f(0,1)\end{aligned}$$

By inspection, $f(0,1) = 0$, so $f\left(\frac{1}{2},1\right) = \sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n} = \ln(2)$. It is easy to compute the desired sum in terms of $f\left(\frac{1}{2},1\right)$, and we find $\sum_{n=1}^{\infty}\frac{1}{2^n(n+4)} = 16\ln(2) - \frac{131}{12}$. Hence, our final answer is $\frac{137}{24} - 8\ln(2)$.

38. Suppose ABC is a triangle with incircle ω, and ω is tangent to $\overline{BC}$ and $\overline{CA}$ at D and E respectively. The bisectors of $\angle A$ and $\angle B$ intersect line DE at F and G respectively, such that $BF = 1$ and $FG = GA = 6$. Compute the radius of ω.

Answer: $\frac{2\sqrt{5}}{5}$
Solution: Let α, β, γ denote the measures of $\frac{1}{2}\angle A, \frac{1}{2}\angle B, \frac{1}{2}\angle C$, respectively. We have $m\angle CEF = 90^\circ - \gamma, m\angle FEA = 90^\circ + \gamma, m\angle AFG = m\angle AFE = 180^\circ - \alpha - (90^\circ + \gamma) = \beta = m\angle ABG$, so $ABFG$ is cyclic. Now $AG = GF$ implies that $\overline{BG}$ bisects $\angle ABF$. Since $\overline{BG}$ by definition bisects $\angle ABC$, we see that F must lie on $\overline{BC}$. Hence, $F = D$. If I denotes the incenter of triangle ABC, then $\overline{ID}$ is perpendicular to $\overline{BC}$, but since A, I, F are collinear, we have that $\overline{AD} \perp \overline{BC}$. Hence, ABC is isoceles with $AB = AC$. Furthermore, $BC = 2BF = 2$. Moreover, since $ABFG$ is cyclic, $\angle BGA$ is a right angle. Construct F' on minor arc GF such that $BF' = 6$ and $F'G = 1$, and let $AB = x$. By

the Pythagorean theorem, $AF' = BG = \sqrt{x^2 - 36}$, so that Ptolemy applied to $ABF'G$ yields $x^2 - 36 = x + 36$. We have $(x-9)(x+8) = 0$. Since x is a length we find $x = 9$. Now we have $AB = AC = 9$. Pythagoras applied to triangle ABD now yields $AD = \sqrt{9^2 - 1^2} = 4\sqrt{5}$, which enables us to compute $[ABC] = \frac{1}{2} \cdot 2 \cdot 4\sqrt{5} = 4\sqrt{5}$. Since the area of a triangle is also equal to its semiperimeter times its inradius, we have $4\sqrt{5} = 10r$ or $r = \frac{2\sqrt{5}}{5}$.

REMARK. In fact, $ABFG$ is always a cyclic quadrilateral for which $\overline{AB}$ plays a diameter. That is, we could have proven this fact without using $FG = GA$.

39. A *fat coin* is one which, when tossed, has a 2/5 probability of being heads, 2/5 of being tails, and 1/5 of landing on its edge. Mr. Fat starts at 0 on the real line. Every minute, he tosses a fat coin. If it's heads, he moves left, decreasing his coordinate by 1; if it's tails, he moves right, increasing his coordinate by 1. If the coin lands on its edge, he moves back to 0. If Mr. Fat does this *ad infinitum*, what fraction of his time will he spend at 0?

Answer: $\frac{1}{3}$
Solution: For $n \in \mathbb{Z}$, let a_n be the fraction of the time Mr. Fat spends at n. By symmetry, $a_n = a_{-n}$ for all n.

For $n > 0$, we have $a_n = \frac{2}{5}a_{n-1} + \frac{2}{5}a_{n+1}$, or $a_{n+1} = \frac{5}{2}a_n - a_{n-1}$. This Fibonacci-like recurrence can be solved explicitly to obtain

$$a_n = \alpha \cdot 2^{|n|} + \beta \cdot 2^{-|n|}$$

for all $n \in \mathbb{Z}$. Now we also have

$$\sum_{n \in \mathbb{Z}} a_n = 1,$$

so we better have $\alpha = 0$, so that $a_0 = \beta$ and $a_{\pm 1} = \frac{\beta}{2}$. Now we also have $a_0 = \frac{2}{5}a_{-1} + \frac{2}{5}a_1 + \frac{1}{5}$, so $\beta = \frac{1}{3}$. This matches perfectly with $\sum_{n \in \mathbb{Z}} a_n = 1$.

40. Compute

$$\sum_{k=1}^{\infty} \frac{3k+1}{2k^3 + k^2} \cdot (-1)^{k+1}.$$

Answer: $\frac{\pi^2}{12} + \frac{\pi}{2} - 2 + \ln 2$
Solution: Via partial fraction decomposition we write the sum as

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{2}{1+2k} + \frac{1}{k^2} \right) (-1)^{k+1}$$

Now recall that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad = \quad S_1$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2) \quad = \quad S_2$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4} \quad = \quad S_3$$

Manipulating (1), we deduce

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right) - 2 \cdot \sum_{k=1}^{\infty} \frac{1}{(2k)^2}$$
$$= \frac{\pi^2}{6} - 2/4 \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12} = S_4$$

It is then easily seen that the answer is equal to $S_2 + 2 \cdot S_3 - 2 + S_4$.

41. Let Γ denote the circumcircle of triangle ABC. Point D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. Points P and Q are on Γ such that $\overline{PQ}$ passes through D and is perpendicular to $\overline{CD}$. Compute PQ, given that $BC = 20, CA = 80, AB = 65$.

Answer: $4\sqrt{745}$
Solution: Suppose that P lies between A and B and Q lies between A and C, and let line PQ intersect lines AC and BC at E and F respectively. As usual, we write a, b, c for the lengths of BC, CA, AB. By the angle bisector theorem, $AD/DB = AC/CB$ so that $AD = \frac{bc}{a+b}$ and $BD = \frac{ac}{a+b}$. Now by Stewart's theorem, $c \cdot CD^2 + \left(\frac{ac}{a+b}\right)\left(\frac{bc}{a+b}\right)c = \frac{a^2bc}{a+b} + \frac{ab^2c}{a+b}$ from which $CD^2 = \frac{ab((a+b)^2-c^2)}{(a+b)^2}$. Now observe that triangles CDE and CDF are congruent, so $ED = DF$. By Menelaus' theorem, $\frac{CA}{AE}\frac{ED}{DF}\frac{FB}{BC} = 1$ so that $\frac{CA}{BC} = \frac{AE}{FB}$. Since $CF = CE$ while $b > a$, it follows that $AE = \frac{b(b-a)}{a+b}$ so that $EC = \frac{2ab}{a+b}$. Finally, $DE = \sqrt{CE^2 - CD^2} = \frac{\sqrt{ab(c^2-(a-b)^2)}}{a+b}$. Plugging in $a = 20, b = 80, c = 65$, we see that $AE = 48, EC = 32, DE = 10$ as well as $AD = 52, BD = 13$. Now let $PD = x, QE = y$. By power of a point about D and E, we have $x(y + 10) = 676$ and $y(x + 10) = 1536$. Subtracting one from the other, we see that $y = x + 86$. Therefore, $x^2 + 96x - 676 = 0$, from which $x = -48 + 2\sqrt{745}$. Finally, $PQ = x + y + 10 = 4\sqrt{745}$.

42. Suppose hypothetically that a certain, very corrupt political entity in a universe holds an election with two candidates, say A and B. A total of 5,825,043 votes are cast, but, in a sudden rainstorm, all the ballots get soaked. Undaunted, the election officials decide to guess what the ballots say. Each ballot has a 51% chance of being deemed a vote for A, and a 49% chance of being deemed a vote for B. The probability that B will win is 10^{-X}. What is X *rounded to the nearest* 10?

Answer: 510
Solution: Let $N = 2912521$, so that the number of ballots cast is $2N + 1$. Let P be the probability that B wins, and let $\alpha = 51\%$ and $\beta = 49\%$ and $\gamma = \beta/\alpha < 1$. We have

$$10^{-X} = P = \sum_{i=0}^{N} \binom{2N+1}{N-i} \alpha^{N-i}\beta^{N+1+i} = \alpha^N \beta^{N+1} \sum_{i=0}^{N} \binom{2N+1}{N-i} \gamma^i$$

(think of $2i + 1$ as representing B's margin of victory). Now

$$\frac{2^{2N+1}}{2N+1} < \binom{2N+1}{N} < \sum_{i=0}^{N} \binom{2N+1}{N-i} \gamma^i < 2^{2N+1},$$

So

$$-X = \log P = N\log\alpha + (N+1)\log\beta + (2N+1)\log 2 - \epsilon = N\log(2\alpha) + (N+1)\log(2\beta) - \epsilon,$$

where $0 < \epsilon < \log(2N+1) < 7$. With a calculator, we find that

$$-X \approx 25048.2 - 25554.2 - \epsilon = -506.0 - \epsilon,$$

so $X \approx 510$.

43. Write down at least one, and up to ten, different 3-digit prime numbers. If you somehow fail to do this, we will ignore your submission for this problem. Otherwise, you're entered into a game with other teams. In this game, you start with 10 points, and each number you write down is like a bet: if no one else writes that number, you gain 1 point, but if anyone else writes that number, you lose 1 point. Thus, your score on this problem can be anything from 0 to 20.

    Solution: There are 143 three-digit primes. None of the following necessarily applies to the actual contest, but it might be useful to think about. Suppose that you're trying to maximize your expected score on this problem. Then you should write down a number if you think the probability that someone else is writing it is less than 1/2 (of course, limit yourself to 10 numbers). You should avoid writting down any number if you think the probability that someone else is writing it is more than 1/2 (of course, write down at least 1 number). Suppose that you expect a total of M different numbers are going to be written down, but have no idea what numbers they might be. If you think $M \geq 72$, you should write down 10 numbers at random; if $M \leq 71$, you should write just 1 number.

44. On the Euclidean plane are given 14 points:

$A = (0, 428)$	$B = (9, 85)$	$C = (42, 865)$	$D = (192, 875)$
$E = (193, 219)$	$F = (204, 108)$	$G = (292, 219)$	$H = (316, 378)$
$I = (375, 688)$	$J = (597, 498)$	$K = (679, 766)$	$L = (739, 641)$
	$M = (772, 307)$	$N = (793, 0)$	

A fly starts at A, visits all the other points, and comes back to A in such a way as to minimize the total distance covered. What path did the fly take? Give the names of the points it visits in order. Your score will be

$$20 + \lfloor\text{the optimal distance}\rfloor - \lfloor\text{your distance}\rfloor$$

or 0, whichever is greater.

Answer: The optimal path is $ACDIKLJMNHGEFB(A)$, or the reverse, of course. In this way the total distance covered by the fly is just over 3591.22.
Solution: This problem is an instance of the Traveling Salesman Problem, which is NP-hard. There is an obvious algorithm in $O(n!)$ time (where n is the number of points), but faster algorithms exist. Nonetheless, the best strategy for solving this problem is probably to draw the points and exercise your geometric intuition.

45. On your answer sheet, *clearly* mark at least seven points, as long as

(i) No three are collinear.

(ii) No seven form a convex heptagon.

Please do not cross out any points; erase if you can do so neatly. If the graders deem that your paper is too messy, or if they determine that you violated one of those conditions, your submission for this problem will be disqualified. Otherwise, your score will be the number of points you marked minus 6, even if you actually violated one of the conditions but were able to fool the graders.

Solution: This is the heptagon case of what is known as the "Happy Ending" or "Erdős-Szekeres" problem, which in general asks, *For any integer* $n \geq 3$*, what is the smallest* $N(n)$*, such that any* $N(n)$ *points in the plane in general position determine a convex* n*-gon?* It is known that such an $N(n)$ always exists and is finite (in fact a specific upper bound has been found). The best known lower bound is $N(n) \geq 2^{n-2}+1$; Erdős and Szekeres conjectured that this bound is tight. The $n \leq 5$ cases have been known for some time. According to the Wikipedia, the $n = 6$ case is solved but unpublished, and for $n \geq 7$, the problem remains open.

For a discussion, see

> W. Morris and V. Soltan. The Erdős-Szekeres Problem on Points in Convex Postion—A Survey, *Bulletin of the American Math Monthly.* **37** (2000), 437–458.

This article is available at

`http://www.ams.org/bull/2000-37-04/S0273-0979-00-00877-6/home.html`.

If $N(7) = 33$, the highest sure score on this problem would be $32 - 6 = 26$. It is not known whether there exist arbitrarily large sets of points that will fool the graders.

The unexamined life is not worth living.