

IXth Annual Harvard-MIT Mathematics Tournament
Saturday 25 February 2006

Team Round B

Robotics [135]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged in an $m \times n$ rectangular grid, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

Our machinery consists of a fleet of identical *mowbots* (or “mobots” for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction, either north or east (*not* south or west). At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered “different” if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot start on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

As an example, one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

1. [25] Prove that the maximum number of mobots you need to mow your lawn is $m + n - 1$.
2. [40] Prove that the minimum number of mobots you need to mow your lawn is $\min\{m, n\}$.
3. [15] Prove that, given any formation, each mobot may be colored in one of three colors — say, white, black, and blue — such that no two adjacent clumps of grass are mowed by different mobots of the same color. Two clumps of grass are adjacent if the distance between them is 1. In your proof, you may use the Four-Color Theorem if you're familiar with it.
4. [15] For $n = m = 4$, find a formation with 6 mobots for which there are exactly 12 ways to color the mobots in three colors as in problem 3. (No proof is necessary.)
5. [40] For $n, m \geq 3$, prove that a formation has exactly six possible colorings satisfying the conditions in problem 3 if and only if there is a mobot that starts at $(1, 1)$.

Polygons [130]

6. [15] Suppose we have a regular hexagon and draw all its sides and diagonals. Into how many regions do the segments divide the hexagon? (No proof is necessary.)
7. [25] Suppose we have an octagon with all angles of 135° , and consecutive sides of alternating length 1 and $\sqrt{2}$. We draw all its sides and diagonals. Into how many regions do the segments divide the octagon? (No proof is necessary.)
8. [25] A regular 12-sided polygon is inscribed in a circle of radius 1. How many chords of the circle that join two of the vertices of the 12-gon have lengths whose squares are rational? (No proof is necessary.)
9. [25] Show a way to construct an equiangular hexagon with side lengths 1, 2, 3, 4, 5, and 6 (not necessarily in that order).
10. [40] Given a convex n -gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal strictly in the interior of the n -gon? Prove that your answer is correct.

What do the following problems have in common? [135]

11. [15] Find the largest positive integer n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square. Prove that your answer is correct.
12. [15] Find all ordered triples (x, y, z) of positive reals such that $x + y + z = 27$ and $x^2 + y^2 + z^2 - xy - yz - zx = 0$. Prove that your answer is correct.
13. [25] Four circles with radii 1, 2, 3, and r are externally tangent to one another. Compute r . (No proof is necessary.)
14. [40] Find the prime factorization of

$$2006^2 \cdot 2262 - 669^2 \cdot 3599 + 1593^2 \cdot 1337.$$

(No proof is necessary.)

15. [40] Let a, b, c, d be real numbers so that c, d are not both 0. Define the function

$$m(x) = \frac{ax + b}{cx + d}$$

on all real numbers x except possibly $-d/c$, in the event that $c \neq 0$. Suppose that the equation $x = m(m(x))$ has at least one solution that is not a solution of $x = m(x)$. Find all possible values of $a + d$. Prove that your answer is correct.