10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

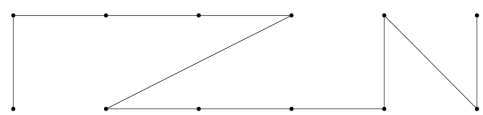
Individual Round: Combinatorics Test

1. [**3**] A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

2. [**3**] How many 5-digit numbers $\overline{abcde}$ exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

3. [**4**] Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

4. [**4**] On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \le i \le 1, 0 \le j \le 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?

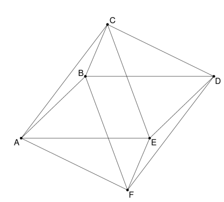


5. [**5**] Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

6. [**5**] Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

7. [**5**] Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off of the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average?

8. [**6**] A set of six edges of a regular octahedron is called *Hamiltonian cycle* if the edges in some order constitute a single continuous loop that visits each vertex exactly once. How many ways are there to partition the twelve edges into two Hamiltonian cycles?



9. [**7**] Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute
$$\sum_{(i,j,k) \in S} ijk.$$

10. [**8**] A subset S of the nonnegative integers is called *supported* if it contains 0, and $k + 8, k + 9 \in S$ for all $k \in S$. How many supported sets are there?