10^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Individual Round: General Test, Part 2

1. [**2**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**2**] A parallelogram has 3 of its vertices at (1,2), (3,8), and (4,1). Compute the sum of all possible x coordinates of the 4th vertex.

3. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

4. [**3**] Three brothers Abel, Banach, and Gauss each have portable music players that can share music with each other. Initially, Abel has 9 songs, Banach has 6 songs, and Gauss has 3 songs, and none of these songs are the same. One day, Abel flips a coin to randomly choose one of his brothers and he adds all of that brother's songs to his collection. The next day, Banach flips a coin to randomly choose one of his brothers and he adds all of that brother's collection of songs to his collection. Finally, each brother randomly plays a song from his collection with each song in his collection being equally likely to be chosen. What is the probability that they all play the same song?

5. [**4**] A best of 9 series is to be played between two teams. That is, the first team to win 5 games is the winner. One of the teams, the Mathletes, has a 2/3 chance of winning any given game. What is the probability that the winner is determined in the 7th game?

6. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

7. [**4**] My friend and I are playing a game with the following rules: If one of us says an integer n, the opponent then says an integer of their choice between $2n$ and $3n$, inclusive. Whoever first says 2007 or greater loses the game, and their opponent wins. I must begin the game by saying a positive integer less than 10. With how many of them can I guarantee a win?

8. [**5**] Compute the number of sequences of numbers $a_1, a_2, \ldots, a_{10}$ such that

   I. $a_i = 0$ or 1 for all i

   II. $a_i \cdot a_{i+1} = 0$ for $i = 1, 2, \ldots, 9$

   III. $a_i \cdot a_{i+2} = 0$ for $i = 1, 2, \ldots, 8$.

9. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,
$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$
Compute the value of $f(2007)$.

10. [**7**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.