10th Annual Harvard-MIT Mathematics Tournament

Saturday 24 February 2007

Individual Round: Geometry Test

1. [**3**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

3. [**4**] Circles ω_1, ω_2, and ω_3 are centered at M, N, and O, respectively. The points of tangency between ω_2 and ω_3, ω_3 and ω_1, and ω_1 and ω_2 are tangent at A, B, and C, respectively. Line MO intersects ω_3 and ω_1 again at P and Q respectively, and line AP intersects ω_2 again at R. Given that ABC is an equilateral triangle of side length 1, compute the area of PQR.

4. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

5. [**5**] Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall. The smallest marble has a radius of 8, and the largest marble has a radius of 18. What is the radius of the middle marble?

6. [**5**] Triangle ABC has $\angle A = 90°$, side $BC = 25, AB > AC$, and area 150. Circle ω is inscribed in ABC, with M its point of tangency on AC. Line BM meets ω a second time at point L. Find the length of segment BL.

7. [**5**] Convex quadrilateral $ABCD$ has sides $AB = BC = 7$, $CD = 5$, and $AD = 3$. Given additionally that $m\angle ABC = 60°$, find BD.

8. [**6**] $ABCD$ is a convex quadrilateral such that $AB < AD$. The diagonal $\overline{AC}$ bisects $\angle BAD$, and $m\angle ABD = 130°$. Let E be a point on the interior of $\overline{AD}$, and $m\angle BAD = 40°$. Given that $BC = CD = DE$, determine $m\angle ACE$ in degrees.

9. [**7**] $\triangle ABC$ is right angled at A. D is a point on AB such that $CD = 1$. AE is the altitude from A to BC. If $BD = BE = 1$, what is the length of AD?

10. [**8**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.