# $11^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

**Saturday 23 February 2008**

**Individual Round: Combinatorics Test**

1. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a $45^\circ$ angle with the horizontal plane.

   **Answer:** $\boxed{60}$ There are 6 such slices, and each slice gives 10 valid pairs (with no overcounting). Therefore, there are 60 such pairs.

2. [**3**] Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of $A$ (when $A$ has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when $A$ is taken over all nonempty subsets of $S$.

   **Answer:** $\boxed{\frac{2009}{2}}$ For any subset $A$, we can define the "reflected subset" $A' = \{i \mid 2009 - i \in A\}$. Then $m(A) = 2009 - m(A')$. Note that as $A$ is taken over all nonempty subsets of $S$, $A'$ goes through all the nonempty subsets of $S$ as well. Thus, the average of $m(A)$ is equal to the average of $\frac{m(A)+m(A')}{2}$, which is the constant $\frac{2009}{2}$.

   *Remark:* : This argument is very analogous to the famous argument that Gauss used to sum the series $1 + 2 + \cdots + 100$.

3. [**4**] Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? Assume that all animals are distinguishable from each other. (Please write your answer as an integer, without any incomplete computations.)

   **Answer:** $\boxed{100800}$ Since there are 9 cow and pigs combined and 7 horses, there must be a pair with 1 cow and 1 pig, and all the other pairs must contain a horse. There are $4 \times 5$ ways of selecting the cow-pig pair, and 7! ways to select the partners for the horses. It follows that the answer is $4 \times 5 \times 7! = 100800$.

4. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

   **Answer:** $\boxed{10201}$

   It is easy to see that the coordinates of the frog's final position must have the same parity. Suppose that the frog went to sleep at $(x, y)$. Then, we have that $-100 \le y \le 100$ and $|x| \le 100 - |y|$, so $x$ can take on the values $-100 + |y|, -98 + |y|, \ldots, 100 - |y|$. There are $101 - |y|$ such values, so the total number of such locations is

   $$\sum_{y=-100}^{100} 101 - |y| = 201 \cdot 101 - 2 \cdot \frac{100(100+1)}{2} = 101^2 = 10201.$$

5. [**5**] Let $S$ be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in $S$ less than 2008.

   **Answer:** $\boxed{128}$ Write the elements of $S$ in their ternary expansion (i.e. base 3). Then the second condition translates into, if $\overline{d_1 d_2 \cdots d_k} \in S$, then $\overline{d_1 d_2 \cdots d_k 0}$ and $\overline{d_1 d_2 \cdots d_k 1}$ are also in $S$. It follows that $S$ is the set of nonnegative integers whose tertiary representation contains only the digits 0 and 1. Since $2 \cdot 3^6 < 2008 < 3^7$, there are $2^7 = 128$ such elements less than 2008. Therefore, there are 128 such non-negative elements.

6. [**5**] A *Sudoku matrix* is defined as a $9 \times 9$ array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine $3 \times 3$ boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

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|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| | 2 | | | | | | | |
| | | | ? | | | | | |
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**Answer:** $\boxed{\frac{2}{21}}$ The third row must contain the digit 1, and it cannot appear in the leftmost three squares. Therefore, the digit 1 must fall into one of the six squares shown below that are marked with $\star$. By symmetry, each starred square has an equal probability of containing the digit 1 (To see this more precisely, note that swapping columns 4 and 5 gives another Sudoku matrix, so the probability that the 4th column $\star$ has the 1 is the same as the probability that the 5th column $\star$ has the 1. Similarly, switching the 4-5-6th columns with the 7-8-9th columns yields another Sudoku matrix, which implies in particular that the probability that the 4th column $\star$ has the 1 is the same as the probability that the 7th column $\star$ has the 1. The rest of the argument follows analogously.) Therefore, the probability that the ? square contains 1 is $1/6$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | |
| | 2 | | | | | | | |
| | | | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ | $\star$ |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
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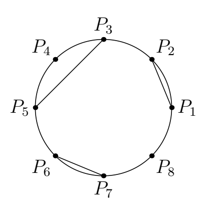
Similarly the probability that the digit 2 appears at ? is also $1/6$. By symmetry, the square ? has equal probability of containing the digits $3, 4, 5, 6, 7, 8, 9$. It follows that this probability is $\left(1 - \frac{1}{6} - \frac{1}{6}\right)/7 = \frac{2}{21}$.

7. [**6**] Let $P_1, P_2, \ldots, P_8$ be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each $P_i$ is the endpoint of at most one segment and (2) two no segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)




**Answer:** $\boxed{323}$ Let $f(n)$ denote the number of valid configurations when there are $n$ points on the circle. Let $P$ be one of the points. If $P$ is not the end point of an edge, then there are $f(n-1)$ ways to connect the remaining $n-1$ points. If $P$ belongs to an edge that separates the circle so that there are $k$ points on one side and $n-k-2$ points on the other side, then there are $f(k)f(n-k-2)$ ways of finishing the configuration. Thus, $f(n)$ satisfies the recurrence relation

$$f(n) = f(n-1) + f(0)f(n-2) + f(1)f(n-3) + f(2)f(n-4) + \cdots + f(n-2)f(0), n \geq 2.$$

The initial conditions are $f(0) = f(1) = 1$. Using the recursion, we find that $f(2) = 2, f(3) = 4, f(4) = 9, f(5) = 21, f(6) = 51, f(7) = 127, f(8) = 323$.

*Remark:* These numbers are known as the *Motzkin numbers.* This is sequence A001006 in the On-Line Encyclopedia of Integer Sequences (`http://www.research.att.com/~njas/sequences/A001006`). In

Richard Stanley's Enumerative Combinatorics Volume 2, one can find 13 different interpretations of Motzkin numbers in exercise 6.38.

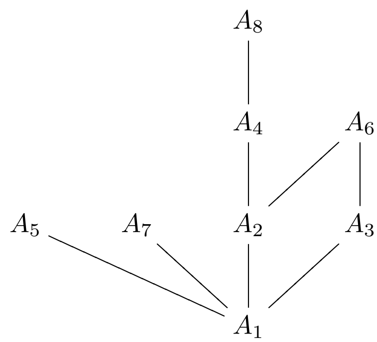
8. [**6**] Determine the number of ways to select a sequence of 8 sets $A_1, A_2, \dots, A_8$, such that each is a subset (possibly empty) of $\{1, 2\}$, and $A_m$ contains $A_n$ if $m$ divides $n$.

**Answer:** $\boxed{2025}$ Consider an arbitrary $x \in \{1, 2\}$, and let us consider the number of ways for $x$ to be in some of the sets so that the constraints are satisfied. We divide into a few cases:

- **Case:** $x \notin A_1$. Then $x$ cannot be in any of the sets. So there is one possibility.
- **Case:** $x \in A_1$ but $x \notin A_2$. Then the only other sets that $x$ could be in are $A_3, A_5, A_7$, and $x$ could be in some collection of them. There are 8 possibilities in this case.
- **Case:** $x \in A_2$. Then $x \in A_1$ automatically. There are 4 independent choices to be make here: (1) whether $x \in A_5$; (2) whether $x \in A_7$; (3) whether $x \in A_3$, and if yes, whether $x \in A_6$; (4) whether $x \in A_4$, and if yes, whether $x \in A_8$. There are $2 \times 2 \times 3 \times 3 = 36$ choices here.

Therefore, there are $1 + 8 + 36 = 45$ ways to place $x$ into some of the sets. Since the choices for $x = 1$ and $x = 2$ are made independently, we see that the total number of possibilities is $45^2 = 2025$.

*Remark:* The solution could be guided by the following diagram. Set $A$ is above $B$ and connected to $B$ if and only if $A \subset B$. Such diagrams are known as *Hasse diagrams*, which are used to depict *partially ordered sets*.




9. [**7**] On an infinite chessboard (whose squares are labeled by $(x, y)$, where $x$ and $y$ range over all integers), a king is placed at $(0, 0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

**Answer:** $\boxed{\frac{1}{4} + \frac{3}{4 \cdot 5^{2008}}}$ Since only the parity of the coordinates are relevant, it is equivalent to consider a situation where the king moves $(1, 0)$ with probability 0.2, moves $(0, 1)$ with probability 0.2, moves $(1, 1)$ with probability 0.2, and stays put with probability 0.4. This can be analyzed using the generating function

$$f(x, y) = (0.4 + 2 \times 0.1x + 2 \times 0.1y + 4 \times 0.05xy)^{2008} = \frac{(2 + x + y + xy)^{2008}}{5^{2008}}.$$

We wish to find the sum of the coefficients of the terms $x^a y^b$, where both $a$ and $b$ are even. This is simply equal to $\frac{1}{4}(f(1, 1) + f(1, -1) + f(-1, 1) + f(-1, -1))$. We have $f(1, 1) = 1$ and $f(1, -1) = f(-1, 1) = f(-1, -1) = 1/5^{2008}$. Therefore, the answer is

$$\frac{1}{4}(f(1, 1) + f(1, -1) + f(-1, 1) + f(-1, -1)) = \frac{1}{4}\left(1 + \frac{3}{5^{2008}}\right) = \frac{1}{4} + \frac{3}{4 \cdot 5^{2008}}.$$

10. [**7**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

**Answer:** $\boxed{1540}$ For each $k = 1, 2, 3, 4$, note that set of pairs $(a_k, b_k)$ with $0 \leq a_k \leq k$ maps bijectively to the set of nonnegative integers through the map $(a_k, b_k) \mapsto a_k + (k+1)b_k$, as $a_k$ is simply the remainder of $a_k + (k+1)b_k$ upon division by $k+1$. By letting $x_k = a_k + (k+1)b_k$, we see that the problem is equivalent to finding the number of quadruples of nonnegative integers $(x_1, x_2, x_3, x_4)$ such that $x_1 + x_2 + x_3 + x_4 = 19$. This is the same as finding the number of quadruples of positive integers $(x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1)$ such that $x_1 + x_2 + x_3 + x_4 = 23$. By a standard "dots and bars" argument, we see that the answer is $\binom{22}{3} = 1540$.

A generating functions solution is also available. It's not hard to see that the answer is the coefficient of $x^{19}$ in

$$\begin{aligned}
&(1+x)\left(1+x+x^2\right)\left(1+x+x^2+x^3\right)\left(1+x+x^2+x^3+x^4\right)\\
&\quad\left(1+x^2+x^4+\cdots\right)\left(1+x^3+x^6+\cdots\right)\left(1+x^4+x^8+\cdots\right)\left(1+x^5+x^{10}+\cdots\right)\\
&=\left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\\
&=\frac{1}{(1-x)^4}=(1-x)^{-4}.
\end{aligned}$$

Using binomial theorem, we find that the coefficient of $x^{19}$ in $(1-x)^{-4}$ is $(-1)^{19}\binom{-4}{19} = \binom{22}{19} = 1540$.