11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Guts Round

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1. [**5**] Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.

2. [**5**] Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. What is the radius of ω?

3. [**5**] How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

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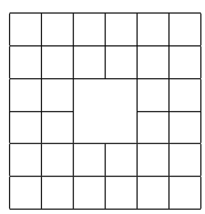
4. [**6**] Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.

5. [**6**] A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?

6. [**6**] Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



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7. [**6**] Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \leq y \leq \pi/2$, find the value of $x + y$.

8. [**6**] Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?

9. [**6**] Consider a circular cone with vertex V, and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV.

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10. [**7**] Find the number of subsets S of $\{1, 2, \ldots 63\}$ the sum of whose elements is 2008.

11. [**7**] Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

12. [**7**] Suppose we have an (infinite) cone $\mathcal{C}$ with apex A and a plane π. The intersection of π and $\mathcal{C}$ is an ellipse $\mathcal{E}$ with major axis BC, such that B is closer to A than C, and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of $\mathcal{C}$ cut up by $\mathcal{E}$ with both spheres tangent to $\mathcal{E}$. What is the ratio of the radii of the spheres (smaller to larger)?

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13. [**8**] Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \ldots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

14. [**8**] Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

15. [**8**] In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

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16. [**9**] Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Find the x-coordinate of the point X on the x-axis maximizing $\angle AXB$.

17. [**9**] Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\ldots + \sqrt{4^{2008}x + 3}}}}} - \sqrt{x} = 1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

18. [**9**] Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135^\circ$, determine BC/AB.

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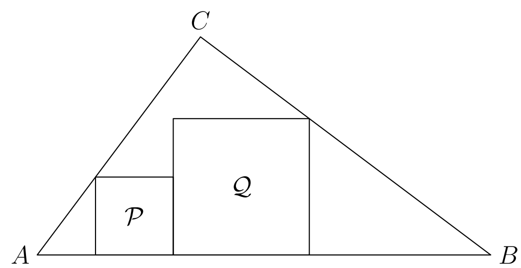
19. [**10**] Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD. Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin \angle PBO$.

20. [**10**] For how many ordered triples (a, b, c) of positive integers are the equations $abc + 9 = ab + bc + ca$ and $a + b + c = 10$ satisfied?

21. [**10**] Let ABC be a triangle with $AB = 5$, $BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Determine the minimum value of the sum of the areas of the two squares.



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22. [**10**] For a positive integer n, let $\theta(n)$ denote the number of integers $0 \leq x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

23. [**10**] Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n, and it tells Jason the number of divisors of n. Then, Kelly and Jason have the following dialogue:

    Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

    Jason: Oh, then I know what the value of n is.

    Kelly: Now I also know what n is.

    Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n?

24. [**10**] Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC.

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25. [**12**] Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

26. [**12**] Let $\mathcal{P}$ be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on $\mathcal{P}$ so that $\angle AV_1B = 90^\circ$. Let $\mathcal{Q}$ be the locus of the midpoint of AB. It turns out that $\mathcal{Q}$ is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio F_1F_2/V_1V_2.

27. [**12**] Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

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28. [**15**] Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

29. [**15**] Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2+y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2+y^2}.$$

Determine the value of $|x| + |y|$.

30. [**15**] Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120^\circ$. Points P and Q lie on segment BC such that

$$AB^2 + BC \cdot CP = BC^2$$
$$3AC^2 + 2BC \cdot CQ = BC^2$$

Find $\angle PAQ$ in degrees.

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31. [**18**] Let $\mathcal{C}$ be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points (P_n) on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing passing through P_n, then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)

    Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

32. [**18**] Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

33. [**18**] Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.

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34. **Who Wants to Be a Millionaire.** In 2000, the Clay Mathematics Institute named seven *Millennium Prize Problems*, with each carrying a prize of \$1 Million for its solution. Write down the name of ONE of the seven Clay Millennium Problems. If your submission is incorrect or misspelled, then your submission is disqualified. If another team wrote down the same Millennium Problem as you, then you get 0 points, otherwise you get 20 points.

35. **NUMB3RS.** The RSA Factoring Challenge, which ended in 2007, challenged computational mathematicians to factor extremely large numbers that were the product of two prime numbers. The largest number successfully factored in this challenge was RSA-640, which has 193 decimal digits and carried a prize of $\$20,000$. The next challenge number carried prize of $\$30,000$, and contains N decimal digits. Your task is to submit a guess for N. Only the team(s) that have the closest guess(es) receives points. If k teams all have the closest guesses, then each of them receives $\lceil \frac{20}{k} \rceil$ points.

36. **The History Channel.** Below is a list of famous mathematicians. Your task is to list a subset of them in the chronological order of their birth dates. Your submission should be a sequence of letters. If your sequence is not in the correct order, then you get 0 points. Otherwise your score will be $\min\{\max\{5(N-4),0\},25\}$, where N is the number of letters in your sequence.

(A) Niels Abel (B) Arthur Cayley (C) Augustus De Morgan (D) Gustav Dirichlet (E) Leonhard Euler (F) Joseph Fourier (G) Évariste Galois (H) Carl Friedrich Gauss (I) Marie-Sophie Germain (J) Joseph Louis Lagrange (K) Pierre-Simon Laplace (L) Henri Poincaré (N) Bernhard Riemann

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