13$^{\text{th}}$Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Algebra Subject Test

1. [**3**] Suppose that x and y are positive reals such that
$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$
Find x.

2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

3. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

4. [**4**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

5. [**5**] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

6. [**5**] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

7. [**5**] Let a, b, c, x, y, and z be complex numbers such that
$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$
If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.

8. [**6**] How many polynomials of degree exactly 5 with real coefficients send the set $\{1, 2, 3, 4, 5, 6\}$ to a permutation of itself?

9. [**7**] Let $f(x) = cx(x-1)$, where c is a positive real number. We use $f^n(x)$ to denote the polynomial obtained by composing f with itself n times. For every positive integer n, all the roots of $f^n(x)$ are real. What is the smallest possible value of c?

10. [**8**] Let $p(x)$ and $q(x)$ be two cubic polynomials such that $p(0) = -24$, $q(0) = 30$, and
$$p(q(x)) = q(p(x))$$
for all real numbers x. Find the ordered pair $(p(3), q(6))$.