13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Algebra Subject Test

1. [**3**] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

Find x.

Answer: $\boxed{\frac{3+\sqrt{17}}{2}}$ Squaring both sides of $x - y^2 = 3$ gives $x^2 + y^4 - 2xy^2 = 9$. Subtract this equation from twice the second given to get $x^2 + 2xy^2 + y^4 = 17 \implies x + y^2 = \pm 17$. Combining this equation with the first given, we see that $x = \frac{3 \pm \sqrt{17}}{2}$. Since x is a positive real, x must be $\frac{3+\sqrt{17}}{2}$.

2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

Answer: $\boxed{(5, 21, 421)}$ Suppose that A and B were rational numbers of rank 3 less than $\frac{1}{4}$, and let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive integers so that $A = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$ and $B = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}$ are the expressions for A and B as stated in the problem. If $b_1 < a_1$ then $A < \frac{1}{a_1 - 1} \leq \frac{1}{b_1} < B$. In other words, of all the rationals less than $\frac{1}{4}$ with rank 3, those that have $a_1 = 5$ are greater than those that have $a_1 = 6, 7, 8, \ldots$ Therefore we can "build" q greedily, adding the largest unit fraction that keeps q less than $\frac{1}{4}$:
$\frac{1}{5}$ is the largest unit fraction less than $\frac{1}{4}$, hence $a_1 = 5$;
$\frac{1}{21}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5}$, hence $a_2 = 21$;
$\frac{1}{421}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5} - \frac{1}{21}$, hence $a_3 = 421$.

3. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

Answer: $\boxed{1092}$ Suppose that $S_N = 0$ for some $N \geq 0$. Then $a_{N+1} = 1$ because $N + 1 \geq S_N$. The following table lists the values of a_k and S_k for a few $k \geq N$:

k	a_k	S_k
N		0
$N+1$	1	$N+1$
$N+2$	1	$2N+3$
$N+3$	-1	N
$N+4$	1	$2N+4$
$N+5$	-1	$N-1$
$N+6$	1	$2N+5$
$N+7$	-1	$N-2$

We see inductively that, for every $i \geq 1$,

$$S_{N+2i} = 2N + 2 + i$$

and

$$S_{N+1+2i} = N + 1 - i$$

thus $S_{3N+3} = 0$ is the next k for which $S_k = 0$. The values of k for which $S_k = 0$ satisfy the recurrence relation $p_{n+1} = 3p_n + 3$, and we compute that the first terms of the sequence are $0, 3, 12, 39, 120, 363, 1092$; hence 1092 is our answer.

4. [**4**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3+bx^2+cx+d=0$ and $bx^3+cx^2+dx+a=0$. Find all possible values of k (including complex values).

Answer: $\boxed{1,-1,i,-i}$ Let k be a root of both polynomials. Multiplying the first polynomial by k and subtracting the second, we have $ak^4-a=0$, which means that k is either 1, -1, i, or $-i$. If $a=b=c=d=1$, then -1, i, and $-i$ are roots of both polynomials. If $a=b=c=1$ and $d=-3$, then 1 is a root of both polynomials. So k can be 1, -1, i, and $-i$.

5. [**5**] Suppose that x and y are complex numbers such that $x+y=1$ and that $x^{20}+y^{20}=20$. Find the sum of all possible values of x^2+y^2.

Answer: $\boxed{-90}$ We have $x^2+y^2+2xy=1$. Define $a=2xy$ and $b=x^2+y^2$ for convenience. Then $a+b=1$ and $b-a=x^2+y^2-2xy=(x-y)^2=2b-1$ so that $x,y=\frac{\sqrt{2b-1}\pm1}{2}$. Then

$$\begin{aligned}
x^{20}+y^{20} &= \left(\frac{\sqrt{2b-1}+1}{2}\right)^{20}+\left(\frac{\sqrt{2b-1}-1}{2}\right)^{20}\\
&= \frac{1}{2^{20}}[(\sqrt{2b-1}+1)^{20}+(\sqrt{2b-1}-1)^{20}]\\
&= \frac{2}{2^{20}}\left[(\sqrt{2b-1})^{20}+\binom{20}{2}(\sqrt{2b-1})^{18}+\binom{20}{4}(\sqrt{2b-1})^{16}+\dots\right]\\
&= \frac{2}{2^{20}}\left[(2b-1)^{10}+\binom{20}{2}(2b-1)^{9}+\binom{20}{4}(2b-1)^{8}+\dots\right]\\
&= 20
\end{aligned}$$

We want to find the sum of distinct roots of the above polynomial in b; we first prove that the original polynomial is square-free. The conditions $x+y=1$ and $x^{20}+y^{20}=20$ imply that $x^{20}+(1-x)^{20}-20=0$; let $p(x)=x^{20}+(1-x)^{20}-20$. p is square-free if and only if $GCD(p,p')=c$ for some constant c:

$$\begin{aligned}
GCD(p,p') &= GCD(x^{20}+(1-x)^{20}-20,20(x^{19}-(1-x)^{19}))\\
&= GCD(x^{20}-x(1-x)^{19}+(1-x)^{19}-20,20(x^{19}-(1-x)^{19}))\\
&= GCD((1-x)^{19}-20,x^{19}-(1-x)^{19})\\
&= GCD((1-x)^{19}-20,x^{19}-20)
\end{aligned}$$

The roots of $x^{19}-20$ are $\sqrt[19]{20^k}\exp(\frac{2\pi ik}{19})$ for some $k=0,1,\dots,18$; the roots of $(1-x)^{19}-20$ are $1-\sqrt[19]{20^k}\exp(\frac{2\pi ik}{19})$ for some $k=0,1,\dots,18$. If $x^{19}-20$ and $(1-x)^{19}-20$ share a common root, then there exist integers m,n such that $\sqrt[19]{20^m}\exp(\frac{2\pi im}{19})=1-\sqrt[19]{20^n}\exp(\frac{2\pi in}{19})$; since the imaginary parts of both sides must be the same, we have $m=n$ and $\sqrt[19]{20^m}\exp(\frac{2\pi im}{19})=\frac{1}{2}\implies 20^m=\frac{1}{2^{19}}$, a contradiction. Thus we have proved that the polynomial in x has no double roots. Since for each b there exists a unique pair (x,y) (up to permutations) that satisfies $x^2+y^2=b$ and $(x+y)^2=1$, the polynomial in b has no double roots.

Let the coefficient of b^n in the above equation be $[b^n]$. By Vieta's Formulas, the sum of all possible values of $b=x^2+y^2$ is equal to $-\frac{[b^9]}{[b^{10}]}$. $[b^{10}]=\frac{2}{2^{20}}\left(2^{10}\right)$ and $[b^9]=\frac{2}{2^{20}}\left(-\binom{10}{1}2^9+\binom{20}{2}2^9\right)$; thus $-\frac{[b^9]}{[b^{10}]}=-\frac{\binom{10}{1}2^9-\binom{20}{2}2^9}{2^{10}}=-90$.

6. [**5**] Suppose that a polynomial of the form $p(x)=x^{2010}\pm x^{2009}\pm\dots\pm x\pm1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

Answer: $\boxed{1005}$ Let $p(x)$ be a polynomial with the maximum number of minus signs.
$p(x)$ cannot have more than 1005 minus signs, otherwise $p(1)<0$ and $p(2)\geq 2^{2010}-2^{2009}-\dots-2-1=1$, which implies, by the Intermediate Value Theorem, that p must have a root greater than 1.
Let $p(x)=\dfrac{x^{2011}+1}{x+1}=x^{2010}-x^{2009}+x^{2008}-\dots-x+1$. -1 is the only real root of $x^{2011}+1=0$ but $p(-1)=2011$; therefore p has no real roots. Since p has 1005 minus signs, it is the desired polynomial.

7. [**5**] Let a, b, c, x, y, and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.

Answer: $\boxed{-5892}$ Manipulate the equations to get a common denominator: $a = \frac{b+c}{x-2} \implies x - 2 = \frac{b+c}{a} \implies x - 1 = \frac{a+b+c}{a} \implies \frac{1}{x-1} = \frac{a}{a+b+c}$; similarly, $\frac{1}{y-1} = \frac{b}{a+b+c}$ and $\frac{1}{z-1} = \frac{c}{a+b+c}$. Thus

$$\begin{aligned}
\frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} &= 1 \\
(y-1)(z-1) + (x-1)(z-1) + (x-1)(y-1) &= (x-1)(y-1)(z-1) \\
xy + yz + zx - 2(x+y+z) + 3 &= xyz - (xy+yz+zx) + (x+y+z) - 1 \\
xyz - 2(xy+yz+zx) + 3(x+y+z) - 4 &= 0 \\
xyz - 2(67) + 3(2010) - 4 &= 0 \\
xyz &= -5892
\end{aligned}$$

8. [**6**] How many polynomials of degree exactly 5 with real coefficients send the set $\{1, 2, 3, 4, 5, 6\}$ to a permutation of itself?

Answer: $\boxed{714}$ For every permutation σ of $\{1, 2, 3, 4, 5, 6\}$, Lagrange Interpolation[1] gives a polynomial of degree at most 5 with $p(x) = \sigma(x)$ for every $x = 1, 2, 3, 4, 5, 6$. Additionally, this polynomial is unique: assume that there exist two polynomials p, q of degree ≤ 5 such that they map $\{1, 2, 3, 4, 5, 6\}$ to the same permutation. Then $p - q$ is a nonzero polynomial of degree ≤ 5 with 6 distinct roots, a contradiction. Thus an upper bound for the answer is $6! = 720$ polynomials.

However, not every polynomial obtained by Lagrange interpolation is of degree 5 (for example, $p(x) = x$). We can count the number of invalid polynomials using finite differences.[2] A polynomial has degree less than 5 if and only if the sequence of 5th finite differences is 0. The 5th finite difference of $p(1), p(2), p(3), p(4), p(5), p(6)$ is $p(1) - 5p(2) + 10p(3) - 10p(4) + 5p(5) - p(6)$; thus we want to solve $p(1) - 5p(2) + 10p(3) - 10p(4) + 5p(5) - p(6) = 0$ with $\{p(1), p(2), p(3), p(4), p(5), p(6)\} = \{1, 2, 3, 4, 5, 6\}$.

Taking the above equation modulo 5, we get $p(1) = p(6) \pmod 5 \implies \{p(1), p(6)\} = \{1, 6\}$. Note that $1 - 5p(2) + 10p(3) - 10p(4) + 5p(5) - 6 = 0$ if and only if $6 - 5p(5) + 10p(4) - 10p(3) + 5p(2) - 1 = 0$, so we may assume that $p(1) = 1$ and double our result later. Then we have $\{p(2), p(3), p(4), p(5)\} = \{2, 3, 4, 5\}$ and

$$-p(2) + 2p(3) - 2p(4) + p(5) = 1.$$

The above equation taken modulo 2 implies that $p(2), p(5)$ are of opposite parity, so $p(3), p(4)$ are of opposite parity. We do casework on $\{p(2), p(5)\}$:

(a) $p(2) = 2, p(5) = 3$; $2p(3) - 2p(4) = 0$ is a contradiction
(b) $p(2) = 2, p(5) = 5$; $2p(3) - 2p(4) = -2 \implies p(3) - p(4) = -1 \implies p(3) = 3, p(4) = 4$
(c) $p(2) = 3, p(5) = 2$; $2p(3) - 2p(4) = -2 \implies p(3) - p(4) = -1 \implies p(3) = 4, p(4) = 5$
(d) $p(2) = 3, p(5) = 4$; $2p(3) - 2p(4) = 0$ is a contradiction
(e) $p(2) = 4, p(5) = 3$; $2p(3) - 2p(4) = 2 \implies p(3) - p(4) = 1$ but $\{p(3), p(4)\} = \{2, 5\}$, contradiction
(f) $p(2) = 4, p(5) = 5$; $2p(3) - 2p(4) = 0$ is a contradiction
(g) $p(2) = 5, p(5) = 2$; $2p(3) - 2p(4) = 4 \implies p(3) - p(4) = 2$, contradiction
(h) $p(2) = 5, p(5) = 4$; $2p(3) - 2p(4) = 2 \implies p(3) - p(4) = 1 \implies p(3) = 3, p(4) = 2$

Hence there are a total of $720 - 2(3) = 714$ polynomials.

[1] See `http://en.wikipedia.org/wiki/Lagrange_interpolation`.
[2] See `http://www.artofproblemsolving.com/Forum/weblog_entry.php?p=1263378`.

9. [**7**] Let $f(x) = cx(x-1)$, where c is a positive real number. We use $f^n(x)$ to denote the polynomial obtained by composing f with itself n times. For every positive integer n, all the roots of $f^n(x)$ are real. What is the smallest possible value of c?

Answer: $\boxed{2}$ We first prove that all roots of $f^n(x)$ are greater than or equal to $-\frac{c}{4}$ and less than or equal to $1+\frac{c}{4}$. Suppose that r is a root of $f^n(x)$. If $r = -\frac{c}{4}$, $f^{-1}(r) = \{\frac{1}{2}\}$ and $-\frac{c}{4} < \frac{1}{2} < 1 + \frac{c}{4}$ since c is positive. Suppose $r \neq -\frac{c}{4}$; by the quadratic formula, there exist two complex numbers r_1, r_2 such that $r_1 + r_2 = 1$ and $f(r_1) = f(r_2) = r$. Thus all the roots of $f^n(x)$ (except $\frac{1}{2}$) come in pairs that sum to 1. No root r of $f^n(x)$ can be less than $-\frac{c}{4}$, otherwise $f^{n+1}(x)$ has an imaginary root, $f^{-1}(r)$. Also, no root r of $f^n(x)$ can be greater than $1+\frac{c}{4}$, otherwise its "conjugate" root will be less than $-\frac{c}{4}$.

Define $g(x) = \frac{1}{2}\left(1+\sqrt{1+\frac{4x}{c}}\right)$, the larger inverse of $f(x)$. Note that $g^n(x)$ is the largest element of $f^{-n}(x)$ (which is a set). $g^n(0)$ should be less than or equal to $1+\frac{c}{4}$ for all n. Let x_0 be the nonzero real number such that $g(x_0) = x_0$; then $cx_0(x_0-1) = x_0 \implies x_o = 1+\frac{1}{c}$. $x_0 < g(x) < x$ if $x > x_0$ and $x < g(x) < x_0$ if $x < x_0$; it can be proved that g^n converges to x_0. Hence we have the requirement that $x_0 = 1+\frac{1}{c} \leq 1+\frac{c}{4} \implies c \geq 2$.

We verify that $c = 2$ is possible. All the roots of $f^-n(x)$ will be real if $g(0) \leq 1+\frac{c}{4} = \frac{3}{2}$. We know that $0 < \frac{3}{2} \implies g(0) < \frac{3}{2}$, so $g^2(0) < \frac{3}{2}$ and $g^n(0) < g^{n+1}(0) < \frac{3}{2}$ for all n. Therefore all the roots of $f^n(x)$ are real.

10. [**8**] Let $p(x)$ and $q(x)$ be two cubic polynomials such that $p(0) = -24$, $q(0) = 30$, and

$$p(q(x)) = q(p(x))$$

for all real numbers x. Find the ordered pair $(p(3), q(6))$.

Answer: $\boxed{(3,-24)}$ Note that the polynomials $f(x) = ax^3$ and $g(x) = -ax^3$ commute under composition. Let $h(x) = x+b$ be a linear polynomial, and note that its inverse $h^{-1}(x) = x-b$ is also a linear polynomial. The composite polynomials $h^{-1}fh$ and $h^{-1}gh$ commute, since function composition is associative, and these polynomials are also cubic.

We solve for the a and b such that $(h^{-1}fh)(0) = -24$ and $(h^{-1}gh)(0) = 30$. We must have:

$$ab^3 - b = -24,\ -ab^3 - b = 30 \Rightarrow a = 1, b = -3$$

These values of a and b yield the polynomials $p(x) = (x-3)^3 + 3$ and $q(x) = -(x-3)^3 + 3$. The polynomials take on the values $p(3) = 3$ and $q(6) = -24$.

Remark: The pair of polynomials found in the solution is not unique. There is, in fact, an entire family of commuting cubic polynomials with $p(0) = -24$ and $q(0) = 30$. They are of the form

$$p(x) = tx(x-3)(x-6) - 24,\ q(x) = -tx(x-3)(x-6) + 30$$

where t is any real number. However, the values of $p(3)$ and $q(6)$ are the same for all polynomials in this family. In fact, if we give the initial conditions $p(0) = k_1$ and $q(0) = k_2$, then we get a general solution of

$$p(x) = t\left(x^3 - \frac{3}{2}(k_1+k_2)x^2 + \frac{1}{2}(k_1+k_2)^2 x\right) + \frac{k_2-k_1}{k_2+k_1}x + k_1$$
$$q(x) = -t\left(x^3 - \frac{3}{2}(k_1+k_2)x^2 + \frac{1}{2}(k_1+k_2)^2 x\right) - \frac{k_2-k_1}{k_2+k_1}x + k_2.$$