13$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Calculus Subject Test

1. [**3**] Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

2. [**3**] Let f be a function such that $f(0) = 1$, $f'(0) = 2$, and
$$f''(t) = 4f'(t) - 3f(t) + 1$$
for all t. Compute the 4th derivative of f, evaluated at 0.

3. [**4**] Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeros of $p'(x)$ are also zeros of $p(x)$. Find p. Note: monic means that the leading coefficient is 1.

4. [**4**] Compute $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} |\cos(k)|}{n}$.

5. [**4**] Let the functions $f(\alpha, x)$ and $g(\alpha)$ be defined as
$$f(\alpha, x) = \frac{\left(\frac{x}{2}\right)^\alpha}{x-1} \qquad\qquad g(\alpha) = \left.\frac{d^4 f}{dx^4}\right|_{x=2}$$
Then $g(\alpha)$ is a polynomial in α. Find the leading coefficient of $g(\alpha)$.

6. [**5**] Let $f(x) = x^3 - x^2$. For a given value of c, the graph of $f(x)$, together with the graph of the line $c + x$, split the plane up into regions. Suppose that c is such that exactly two of these regions have finite area. Find the value of c that minimizes the sum of the areas of these two regions.

7. [**6**] Let a_1, a_2, and a_3 be nonzero complex numbers with non-negative real and imaginary parts. Find the minimum possible value of
$$\frac{|a_1 + a_2 + a_3|}{\sqrt[3]{|a_1 a_2 a_3|}}.$$

8. [**6**] Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Calculate $\sum_{n=2}^{\infty} f(n)$.

9. [**7**] Let $x(t)$ be a solution to the differential equation
$$\left(x + x'\right)^2 + x \cdot x'' = \cos t$$
with $x(0) = x'(0) = \sqrt{\frac{2}{5}}$. Compute $x\left(\frac{\pi}{4}\right)$.

10. [**8**] Let $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. Then there exists constants γ, c, and d such that
$$f(n) = \ln(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + O\left(\frac{1}{n^3}\right),$$
where the $O(\frac{1}{n^3})$ means terms of order $\frac{1}{n^3}$ or lower. Compute the ordered pair (c, d).