13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Calculus Subject Test

1. [**3**] Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

   Answer: $\boxed{50}$ Observe that $p(x)$ must be quadratic. Let $p(x) = ax^2 + bx + c$. Comparing coefficients gives $a = 1$, $b - 2a = 2$, and $c - b = 1$. So $b = 4$, $c = 5$, $p(x) = x^2 + 4x + 5$ and $p(5) = 25 + 20 + 5 = 50$.

2. [**3**] Let f be a function such that $f(0) = 1$, $f'(0) = 2$, and

$$f''(t) = 4f'(t) - 3f(t) + 1$$

   for all t. Compute the 4th derivative of f, evaluated at 0.

   Answer: $\boxed{54}$ Putting $t = 0$ gives $f''(0) = 6$. By differentiating both sides, we get $f^{(3)}(t) = 4f''(t) - 3f'(t)$ and $f^{(3)}(0) = 4 \cdot 6 - 3 \cdot 2 = 18$. Similarly, $f^{(4)}(t) = 4f^{(3)}(t) - 3f''(t)$ and $f^{(4)}(0) = 4 \cdot 18 - 3 \cdot 6 = 54$.

3. [**4**] Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeros of $p'(x)$ are also zeros of $p(x)$. Find p. Note: monic means that the leading coefficient is 1.

   Answer: $\boxed{(x+1)^3}$ A root of a polynomial p will be a double root if and only if it is also a root of p'. Let a and b be the roots of p'. Since a and b are also roots of p, they are double roots of p. But p can have only three roots, so $a = b$ and a becomes a double root of p'. This makes $p'(x) = 3c(x - a)^2$ for some constant $3c$, and thus $p(x) = c(x - a)^3 + d$. Because a is a root of p and p is monic, $d = 0$ and $c = 1$. From $p(0) = 1$ we get $p(x) = (x + 1)^3$.

4. [**4**] Compute $\lim\limits_{n\to\infty} \dfrac{\sum_{k=1}^{n} |\cos(k)|}{n}$.

   Answer: $\boxed{\frac{2}{\pi}}$ The main idea lies on the fact that positive integers are uniformly distributed modulo π. (In the other words, if each integer n is written as $q\pi + r$ where q is an integer and $0 \le r < \pi$, the value of r will distribute uniformly in the interval $[0, \pi]$.) Using this fact, the summation is equivalent to the average value (using the Riemann summation) of the function $|\cos(k)|$ over the interval $[0, \pi]$. Therefore, the answer is $\frac{1}{\pi}\int_0^{\pi} |\cos(k)| = \frac{2}{\pi}$.

5. [**4**] Let the functions $f(\alpha, x)$ and $g(\alpha)$ be defined as

$$f(\alpha, x) = \frac{\left(\frac{x}{2}\right)^{\alpha}}{x - 1} \qquad\qquad g(\alpha) = \left.\frac{d^4 f}{dx^4}\right|_{x=2}$$

   Then $g(\alpha)$ is a polynomial in α. Find the leading coefficient of $g(\alpha)$.

   Answer: $\boxed{\frac{1}{16}}$ Write the first equation as $(x-1)f = \left(\frac{x}{2}\right)^{\alpha}$. For now, treat α as a constant. From this equation, repeatedly applying derivative with respect to x gives

$$\begin{aligned}
(x-1)f' + f &= \left(\frac{\alpha}{2}\right)\left(\frac{x}{2}\right)^{\alpha-1} \\
(x-1)f'' + 2f' &= \left(\frac{\alpha}{2}\right)\left(\frac{\alpha-1}{2}\right)\left(\frac{x}{2}\right)^{\alpha-2} \\
(x-1)f^{(3)} + 3f'' &= \left(\frac{\alpha}{2}\right)\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha-2}{2}\right)\left(\frac{x}{2}\right)^{\alpha-3} \\
(x-1)f^{(4)} + 4f^{(3)} &= \left(\frac{\alpha}{2}\right)\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha-2}{2}\right)\left(\frac{\alpha-3}{2}\right)\left(\frac{x}{2}\right)^{\alpha-4}
\end{aligned}$$

   Substituting $x = 2$ to all equations gives $g(\alpha) = f^{(4)}(\alpha, 2) = \left(\frac{\alpha}{2}\right)\left(\frac{\alpha-1}{2}\right)\left(\frac{\alpha-2}{2}\right)\left(\frac{\alpha-3}{2}\right) - 4f^{(3)}(\alpha, 2)$. Because $f^{(3)}(\alpha, 2)$ is a cubic polynomial in α, the leading coefficient of $g(\alpha)$ is $\frac{1}{16}$.

6. [**5**] Let $f(x) = x^3 - x^2$. For a given value of c, the graph of $f(x)$, together with the graph of the line $c + x$, split the plane up into regions. Suppose that c is such that exactly two of these regions have finite area. Find the value of c that minimizes the sum of the areas of these two regions.

   Answer: $\boxed{-\frac{11}{27}}$ Observe that $f(x)$ can be written as $(x - \frac{1}{3})^3 - \frac{1}{3}(x - \frac{1}{3}) - \frac{2}{27}$, which has $180°$ symmetry around the point $(\frac{1}{3}, -\frac{2}{27})$. Suppose the graph of f cuts the line $y = c + x$ into two segments of lengths a and b. When we move the line toward point P with a small distance Δx (measured along the line perpendicular to $y = x + c$), the sum of the enclosed areas will increase by $|a - b|(\Delta x)$. As long as the line $x + c$ does not passes through P, we can find a new line $x + c^*$ that increases the sum of the enclosed areas. Therefore, the sum of the areas reaches its maximum when the line passes through P. For that line, we can find that $c = y - x = -\frac{2}{27} - \frac{1}{3} = -\frac{11}{27}$.

7. [**6**] Let a_1, a_2, and a_3 be nonzero complex numbers with non-negative real and imaginary parts. Find the minimum possible value of

$$\frac{|a_1 + a_2 + a_3|}{\sqrt[3]{|a_1 a_2 a_3|}}.$$

   Answer: $\boxed{\sqrt{3}\sqrt[3]{2}}$ Write a_1 in its polar form $re^{i\theta}$ where $0 \le \theta \le \frac{\pi}{2}$. Suppose a_2, a_3 and r are fixed so that the denominator is constant. Write $a_2 + a_3$ as $se^{i\phi}$. Since a_2 and a_3 have non-negative real and imaginary parts, the angle ϕ lies between 0 and $\frac{\pi}{2}$. Consider the function

$$f(\theta) = |a_1 + a_2 + a_3|^2 = |re^{i\theta} + se^{i\phi}|^2 = r^2 + 2rs\cos(\theta - \phi) + s^2.$$

   Its second derivative is $f''(\theta) = -2rs(\cos(\theta - \phi)))$. Since $-\frac{\pi}{2} \le (\theta - \phi) \le \frac{\pi}{2}$, we know that $f''(\theta) < 0$ and f is concave. Therefore, to minimize f, the angle θ must be either 0 or $\frac{\pi}{2}$. Similarly, each of a_1, a_2 and a_3 must be either purely real or purely imaginary to minimize f and the original fraction.

   By the AM-GM inequality, if a_1, a_2 and a_3 are all real or all imaginary, then the minimum value of the fraction is 3. Now suppose only two of the a_i's, say, a_1 and a_2 are real. Since the fraction is homogenous, we may fix $a_1 + a_2$ - let the sum be 2. The term $a_1 a_2$ in the denominator acheives its maximum only when a_1 and a_2 are equal, i.e. when $a_1 = a_2 = 1$. Then, if $a_3 = ki$ for some real number k, then the expression equals

$$\frac{\sqrt{k^2 + 4}}{\sqrt[3]{k}}.$$

   Squaring and taking the derivative, we find that the minimum value of the fraction is $\sqrt{3}\sqrt[3]{2}$, attained when $k = \sqrt{2}$. With similar reasoning, the case where only one of the a_i's is real yields the same minimum value.

8. [**6**] Let $f(n) = \displaystyle\sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Calculate $\displaystyle\sum_{n=2}^{\infty} f(n)$.

   Answer: $\boxed{3 - e}$

$$\begin{aligned}
\sum_{n=2}^{\infty} f(n) &= \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{k^n \cdot k!} \\
&= \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{n=2}^{\infty} \frac{1}{k^n} \\
&= \sum_{k=2}^{\infty} \frac{1}{k!} \cdot \frac{1}{k(k-1)} \\
&= \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \cdot \frac{1}{k^2(k-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\left(\frac{1}{k-1} - \frac{1}{k^2} - \frac{1}{k}\right) \\
&= \sum_{k=2}^{\infty}\left(\frac{1}{(k-1)(k-1)!} - \frac{1}{k\cdot k!} - \frac{1}{k!}\right) \\
&= \sum_{k=2}^{\infty}\left(\frac{1}{(k-1)(k-1)!} - \frac{1}{k\cdot k!}\right) - \sum_{k=2}^{\infty}\frac{1}{k!} \\
&= \frac{1}{1\cdot 1!} - \left(e - \frac{1}{0!} - \frac{1}{1!}\right) \\
&= 3 - e
\end{aligned}
$$

9. [**7**] Let $x(t)$ be a solution to the differential equation

$$(x + x')^2 + x \cdot x'' = \cos t$$

with $x(0) = x'(0) = \sqrt{\frac{2}{5}}$. Compute $x\left(\frac{\pi}{4}\right)$.

Answer: $\boxed{\frac{\sqrt[4]{450}}{5}}$ Rewrite the equation as $x^2 + 2xx' + (xx')' = \cos t$. Let $y = x^2$, so $y' = 2xx'$ and the equation becomes $y + y' + \frac{1}{2}y'' = \cos t$. The term $\cos t$ suggests that the particular solution should be in the form $A\sin t + B\cos t$. By substitution and coefficient comparison, we get $A = \frac{4}{5}$ and $B = \frac{2}{5}$. Since the function $y(t) = \frac{4}{5}\sin t + \frac{2}{5}\cos t$ already satisfies the initial conditions $y(0) = x(0)^2 = \frac{2}{5}$ and $y'(0) = 2x(0)x'(0) = \frac{4}{5}$, the function y also solves the initial value problem. Note that since x is positive at $t = 0$ and $y = x^2$ never reaches zero before t reaches $\frac{\pi}{4}$, the value of $x\left(\frac{\pi}{4}\right)$ must be positive. Therefore, $x\left(\frac{\pi}{4}\right) = +\sqrt{y\left(\frac{\pi}{4}\right)} = \sqrt{\frac{6}{5}\cdot\frac{\sqrt{2}}{2}} = \frac{\sqrt[4]{450}}{5}$.

10. [**8**] Let $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. Then there exists constants γ, c, and d such that

$$f(n) = \ln(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + O(\frac{1}{n^3}),$$

where the $O(\frac{1}{n^3})$ means terms of order $\frac{1}{n^3}$ or lower. Compute the ordered pair (c, d).

Answer: $\boxed{(\frac{1}{2}, -\frac{1}{12})}$ From the given formula, we pull out the term $\frac{k}{n^3}$ from $O(\frac{1}{n^4})$, making $f(n) = \log(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + \frac{k}{n^3} + O(\frac{1}{n^4})$. Therefore,

$$f(n+1) - f(n) = \log\left(\frac{n+1}{n}\right) - c\left(\frac{1}{n} - \frac{1}{n+1}\right) - d\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - k\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + O\left(\frac{1}{n^4}\right).$$

For the left hand side, $f(n+1) - f(n) = \frac{1}{n+1}$. By substituting $x = \frac{1}{n}$, the formula above becomes

$$\frac{x}{x+1} = \log(1+x) - cx^2 \cdot \frac{1}{x+1} - dx^3 \cdot \frac{x+2}{(x+1)^2} - kx^4 \cdot \frac{x^2+3x+3}{(x+1)^3} + O(x^4).$$

Because x is on the order of $\frac{1}{n}$, $\frac{1}{(x+1)^3}$ is on the order of a constant. Therefore, all the terms in the expansion of $kx^4 \cdot \frac{x^2+3x+3}{(x+1)^3}$ are of order x^4 or higher, so we can collapse it into $O(x^4)$. Using the Taylor expansions, we get

$$x\left(1 - x + x^2\right) + O\left(x^4\right) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) - cx^2(1-x) - dx^3(2) + O\left(x^4\right).$$

Coefficient comparison gives $c = \frac{1}{2}$ and $d = -\frac{1}{12}$.