13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

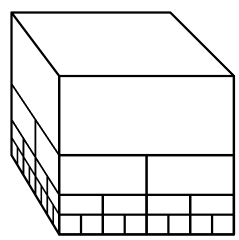
Combinatorics Subject Test

1. [**2**] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

2. [**3**] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

3. [**4**] How many ways are there to choose 2010 functions $f_1, \ldots, f_{2010}$ from $\{0, 1\}$ to $\{0, 1\}$ such that $f_{2010} \circ f_{2009} \circ \cdots \circ f_1$ is constant? Note: a function g is constant if $g(a) = g(b)$ for all a, b in the domain of g.

4. [**4**] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the kth layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



5. [**5**] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

6. [**5**] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y), it can move to $(x - 1, y - 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$, or $(x + 1, y + 1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

7. [**6**] For each integer x with $1 \le x \le 10$, a point is randomly placed at either $(x, 1)$ or $(x, -1)$ with equal probability. What is the expected area of the convex hull of these points? Note: the convex hull of a finite set is the smallest convex polygon containing it.

8. [**6**] How many functions f from $\{-1005, \ldots, 1005\}$ to $\{-2010, \ldots, 2010\}$ are there such that the following two conditions are satisfied?

   - If $a < b$ then $f(a) < f(b)$.
   - There is no n in $\{-1005, \ldots, 1005\}$ such that $|f(n)| = |n|$.

9. [**7**] Rosencrantz and Guildenstern are playing a game where they repeatedly flip coins. Rosencrantz wins if 1 heads followed by 2009 tails appears. Guildenstern wins if 2010 heads come in a row. They will flip coins until someone wins. What is the probability that Rosencrantz wins?

10. [**8**] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?