13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

General Test, Part 1

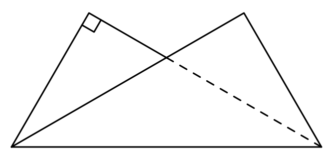
1. [**3**] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

Find x.

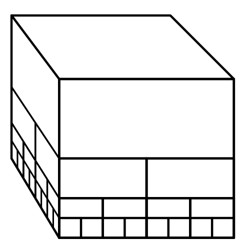
2. [**3**] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

3. [**4**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



4. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

5. [**4**] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the kth layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



6. [**5**] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

7. [**6**] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

8. [**6**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

9. [**7**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

10. [**8**] Let a, b, c, x, y, and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.