13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

General Test, Part 1

1. [**3**] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

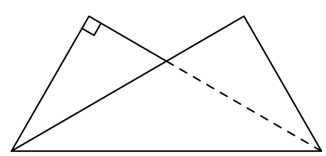
Find x.

Answer: $\boxed{\frac{3+\sqrt{17}}{2}}$ Squaring both sides of $x - y^2 = 3$ gives $x^2 + y^4 - 2xy^2 = 9$. Subtract this equation from twice the second given to get $x^2 + 2xy^2 + y^4 = 17 \implies x + y^2 = \pm 17$. Combining this equation with the first given, we see that $x = \frac{3 \pm \sqrt{17}}{2}$. Since x is a positive real, x must be $\frac{3+\sqrt{17}}{2}$.

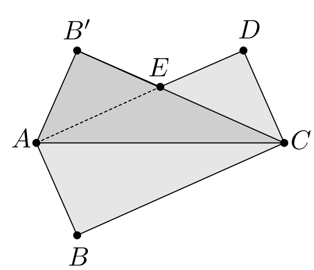
2. [**3**] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

Answer: $\boxed{180}$ We partition the elements of S into the following subsets: $\{1, 2, 4, 8\}$, $\{3, 6\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$. Consider the first subset, $\{1, 2, 4, 8\}$. Say 2 is an element of T. Because $2 \cdot 2 = 4$ is in S, 4 must also be in T. Furthermore, since $4 \cdot 2 = 8$ is in S, 8 must also be in T. So if T contains 2, it must also contain 4 and 8. Similarly, if T contains 1, it must also contain 2, 4, and 8. So T can contain the following subsets of the subset $\{1, 2, 4, 8\}$: the empty set, $\{8\}$, $\{4, 8\}$, $\{2, 4, 8\}$, or $\{1, 2, 4, 8\}$. This gives 5 possibilities for the first subset. In general, we see that if T contains an element q of one of these subsets, it must also contain the elements in that subset that are larger than q, because we created the subsets for this to be true. So there are 3 possibilities for $\{3, 6\}$, 3 for $\{5, 10\}$, 2 for $\{7\}$, and 2 for $\{9\}$. This gives a total of $5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 180$ possible subsets T.

3. [**4**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



Answer: $\boxed{\sqrt{5}}$



Given a polygon $P_1P_2 \cdots P_k$, let $[P_1P_2 \cdots P_k]$ denote its area. Let $ABCD$ be the rectangle. Suppose we fold B across $\overline{AC}$, and let E be the intersection of $\overline{AD}$ and $\overline{B'C}$. Then we end up with the pentagon $ACDEB'$, depicted above. Let's suppose, without loss of generality, that $ABCD$ has area 1. Then $\triangle AEC$ must have area $\frac{3}{10}$, since

$$\begin{aligned}[ABCD] &= [ABC] + [ACD] \\ &= [AB'C] + [ACD] \\ &= [AB'E] + 2[AEC] + [EDC] \\ &= [ACDEB'] + [AEC] \\ &= \frac{7}{10}[ABCD] + [AEC],\end{aligned}$$

That is, $[AEC] = \frac{3}{10}[ABCD] = \frac{3}{10}$.

Since $\triangle ECD$ is congruent to $\triangle EAB'$, both triangles have area $\frac{1}{5}$. Note that $\triangle AB'C$, $\triangle ABC$, and $\triangle CDA$ are all congruent, and all have area $\frac{1}{2}$. Since $\triangle AEC$ and $\triangle EDC$ share altitude $\overline{DC}$, $\frac{DE}{EA} = \frac{[DEC]}{[AEC]} = \frac{2}{3}$. Because $\triangle CAE$ is isosceles, $CE = EA$. Let $AE = 3x$. The $CE = 3x$, $DE = 2x$, and $CD = x\sqrt{9-4} = x\sqrt{5}$. Then $\frac{AD}{DC} = \frac{AE+ED}{DC} = \frac{3+2}{\sqrt{5}} = \sqrt{5}$.

4. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

Answer: $\boxed{1092}$ Suppose that $S_N = 0$ for some $N \geq 0$. Then $a_{N+1} = 1$ because $N+1 \geq S_N$. The following table lists the values of a_k and S_k for a few $k \geq N$:

k	a_k	S_k
N		0
$N+1$	1	$N+1$
$N+2$	1	$2N+3$
$N+3$	-1	N
$N+4$	1	$2N+4$
$N+5$	-1	$N-1$
$N+6$	1	$2N+5$
$N+7$	-1	$N-2$

We see inductively that, for every $i \geq 1$,

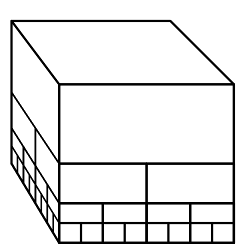
$$S_{N+2i} = 2N + 2 + i$$

and

$$S_{N+1+2i} = N + 1 - i$$

thus $S_{3N+3} = 0$ is the next k for which $S_k = 0$. The values of k for which $S_k = 0$ satisfy the recurrence relation $p_{n+1} = 3p_n + 3$, and we compute that the first terms of the sequence are $0, 3, 12, 39, 120, 363, 1092$; hence 1092 is our answer.

5. [**4**] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the kth layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



Answer: $\boxed{3384}$ Each time Laura removes a block, 4 additional blocks are exposed, increasing the total number of exposed blocks by 3. She removes 5 blocks, for a total of $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13$ ways. However, the stack originally only has 4 layers, so we must subtract the cases where removing a block on the bottom layer does not expose any new blocks. There are $1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 256$ of these (the last factor of 4 is from the 4 blocks that we counted as being exposed, but were not actually). So our final answer is $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 - 1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 3384$.

6. [**5**] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

   Answer: $\boxed{20503}$

   Let the number of 2, 5, and 10 dollar notes John can use be x, y, and z respectively. We wish to find the number of nonnegative integer solutions to $2x + 5y + 10z = 2010$. Consider this equation mod 2. Because $2x$, $10z$, and 2010 are even, $5y$ must also be even, so y must be even. Now consider the equation mod 5. Because $5y$, $10z$, and 2010 are divisible by 5, $2x$ must also be divisible by 5, so x must be divisible by 5. So both $2x$ and $5y$ are divisible by 10. So the equation is equivalent to $10x' + 10y' + 10z = 2010$, or $x' + y' + z = 201$, with x', y', and z nonnegative integers. There is a well-known bijection between solutions of this equation and picking 2 of 203 balls in a row on the table (explained in further detail below), so there are $\binom{203}{2} = 20503$ ways.

   The bijection between solutions of $x' + y' + z = 201$ and arrangements of 203 balls in a row is as follows. Given a solution of the equation, we put x' white balls in a row, then a black ball, then y' white balls, then a black ball, then z white balls. This is like having 203 balls in a row on a table and picking two of them to be black. To go from an arrangement of balls to a solution of the equation, we just read off x', y', and z from the number of white balls in a row. There are $\binom{203}{2}$ ways to choose 2 of 203 balls to be black, so there are $\binom{203}{2}$ solutions to $x' + y' + z = 201$.

7. [**6**] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

   Answer: $\boxed{1005}$ Let $p(x)$ be a polynomial with the maximum number of minus signs.
   $p(x)$ cannot have more than 1005 minus signs, otherwise $p(1) < 0$ and $p(2) \geq 2^{2010} - 2^{2009} - \ldots - 2 - 1 = 1$, which implies, by the Intermediate Value Theorem, that p must have a root greater than 1.
   Let $p(x) = \dfrac{x^{2011} + 1}{x + 1} = x^{2010} - x^{2009} + x^{2008} - \ldots - x + 1$. -1 is the only real root of $x^{2011} + 1 = 0$ but $p(-1) = 2011$; therefore p has no real roots. Since p has 1005 minus signs, it is the desired polynomial.

8. [**6**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

   Answer: $\boxed{2}$ The answer is 2 for any number of dimensions. We prove this by induction on the dimension.

   Note that 1-dimensional spheres are pairs of points, and 2-dimensional spheres are circles.

   Base case, $d = 2$: The intersection of two circles is either a circle (if the original circles are identical, and in the same place), a pair of points, a single point (if the circles are tangent), or the empty set. Thus, in dimension 2, the largest finite number of intersection points is 2, because the number of pairwise intersection points is 0, 1, or 2 for distinct circles.

   We now prove that the intersection of two k-dimensional spheres is either the empty set, a $(k-1)$-dimensional sphere, a k-dimensional sphere (which only occurs if the original spheres are identical and coincident). Consider two spheres in k-dimensional space, and impose a coordinate system such that the centers of the two spheres lie on one coordinate axis. Then the equations for the two spheres become identical in all but one coordinate:

$$\begin{aligned}(x_1 - a_1)^2 + x_2^2 + \ldots + x_k^2 &= r_1^2 \\ (x_1 - a_2)^2 + x_2^2 + \ldots + x_k^2 &= r_2^2\end{aligned}$$

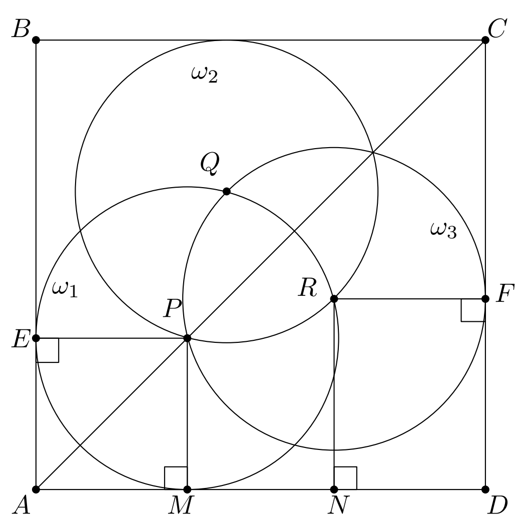
If $a_1 = a_2$, the spheres are concentric, and so they are either nonintersecting or coincident, intersecting in a k-dimensional sphere. If $a_1 \neq a_2$, then subtracting the equations and solving for x_1 yields $x_1 = \frac{r_1^2 - a_1^2 - r_2^2 + a_2^2}{2(a_2 - a_2)}$. Plugging this in to either equation above yields a single equation equation that describes a $(k-1)$-dimensional sphere.

Assume we are in dimension d, and suppose for induction that for all k less than d, any two distinct k-dimensional spheres intersecting in a finite number of points intersect in at most two points. Suppose we have a collection of d-dimensional spheres $s_1, s_2, \ldots, s_m$. Without loss of generality, suppose the s_i are distinct. Let t_i be the intersection of s_i and s_{i+1} for $1 \leq i < m$. If any t_i are the empty set, then the intersection of the t_i is empty. None of the t_i is a d-dimensional sphere because the s_i are distinct. Thus each of $t_1, t_2, \ldots, t_{m-1}$ is a $(d-1)$-dimensional sphere, and the intersection of all of them is the same as the intersection of the d-dimensional spheres. We can then apply the inductive hypothesis to find that $t_1, \ldots, t_{m-1}$ intersect in at most two points. Thus, by induction, a set of spheres in any dimension which intersect at only finitely many points intersect at at most two points.

We now exhibit a set of 2^{2009} 2010-dimensional spheres, and prove that their intersection contains exactly two points. Take the spheres with radii $\sqrt{2013}$ and centers $(0, \pm 1, \pm 1, \ldots, \pm 1)$, where the sign of each coordinate is independent from the sign of every other coordinate. Because of our choice of radius, all these spheres pass through the points $(\pm 2, 0, 0, \ldots 0)$. Then the intersection is the set of points $(x_1, x_2, \ldots, x_{2010})$ which satisfy the equations $x_1^2 + (x_2 \pm 1)^2 + \cdots + (x_{2010} \pm 1)^2 = 2013$. The only solutions to these equations are the points $(\pm 2, 0, 0, \ldots, 0)$ (since $(x_i + 1)^2$ must be the same as $(x_i - 1)^2$ for all $i > 1$, because we may hold all but one of the $\pm$ choices constant, and change the remaining one). $\square$

9. [**7**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

Answer: $\boxed{\frac{\sqrt{6}+\sqrt{2}+8}{4}}$



By the Pigeonhole Principle, two of the sides must be tangent to the same circle, say ω_1. Since S surrounds the circles, these two sides must be adjacent, so we can let A denote the common vertex of the two sides tangent to ω_1. Let B, C, and D be the other vertices of S in clockwise order, and let P, Q, and R be the centers of ω_1, ω_2, and ω_3 respectively, and suppose WLOG that they are also in clockwise order. Then AC passes through the center of ω_1, and by symmetry (since $AB = AD$) it must also pass through the other intersection point of ω_2 and ω_3. That is, AC is the radical axis of ω_2 and ω_3.

Now, let M and N be the feet of the perpendiculars from P and R, respectively, to side AD. Let E and F be the feet of the perpendiculars from P to AB and from R to DC, respectively. Then $PEAM$ and $NRFD$ are rectangles, and PE and RF are radii of ω_1 and ω_2 respectively. Thus $AM = EP = 1$ and $ND = RF = 1$. Finally, we have

$$\begin{aligned}
MN &= PR \cdot \cos(180^\circ - \angle EPR) \\
&= \cos(180^\circ - EPQ - RPQ) \\
&= -\cos\left((270^\circ - 60^\circ)/2 + 60^\circ\right) \\
&= -\cos(165^\circ) \\
&= \cos(15^\circ) \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}.
\end{aligned}$$

Thus $AD = AM + MN + ND = 1 + \frac{\sqrt{6}+\sqrt{2}}{4} + 1 = \frac{\sqrt{6}+\sqrt{2}+8}{4}$ as claimed. □

10. [**8**] Let a, b, c, x, y, and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.

Answer: $\boxed{-5892}$ Manipulate the equations to get a common denominator: $a = \frac{b+c}{x-2} \implies x - 2 = \frac{b+c}{a} \implies x - 1 = \frac{a+b+c}{a} \implies \frac{1}{x-1} = \frac{a}{a+b+c}$; similarly, $\frac{1}{y-1} = \frac{b}{a+b+c}$ and $\frac{1}{z-1} = \frac{c}{a+b+c}$. Thus

$$\begin{aligned}
\frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} &= 1 \\
(y-1)(z-1) + (x-1)(z-1) + (x-1)(y-1) &= (x-1)(y-1)(z-1) \\
xy + yz + zx - 2(x+y+z) + 3 &= xyz - (xy+yz+zx) + (x+y+z) - 1 \\
xyz - 2(xy+yz+zx) + 3(x+y+z) - 4 &= 0 \\
xyz - 2(67) + 3(2010) - 4 &= 0 \\
xyz &= -5892
\end{aligned}$$