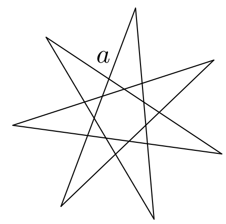
13^{th}Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

General Test, Part 2

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

3. [**4**] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

4. [**4**] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

5. [**5**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

6. [**5**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

7. [**5**] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

8. [**6**] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y), it can move to $(x - 1, y - 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$, or $(x + 1, y + 1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

9. [**7**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

10. [**8**] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?