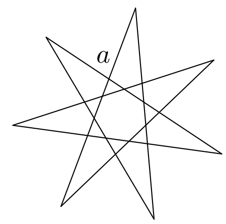
13^{th} Annual Harvard-MIT Mathematics Tournament

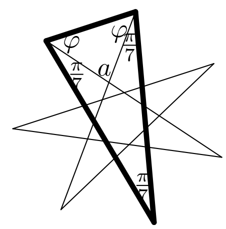
Saturday 20 February 2010

General Test, Part 2

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



Answer: $\boxed{\frac{3\pi}{7}}$ The measure of the interior angle of a point of the star is $\frac{\pi}{7}$ because it is an inscribed angle on the circumcircle which intercepts a seventh of the circle.[1]



Consider the triangle shown above in bold. Because the sum of the angles in any triangle is π,

$$2\varphi + 3\left(\frac{\pi}{7}\right) = \pi = 2\varphi + a$$

Canceling the 2φ on the right-hand side and on the left-hand side, we obtain

$$a = \frac{3\pi}{7}.$$

□

2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

Answer: $\boxed{(5, 21, 421)}$ Suppose that A and B were rational numbers of rank 3 less than $\frac{1}{4}$, and let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive integers so that $A = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$ and $B = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}$ are the expressions for A and B as stated in the problem. If $b_1 < a_1$ then $A < \frac{1}{a_1 - 1} \leq \frac{1}{b_1} < B$. In other words, of all the rationals less than $\frac{1}{4}$ with rank 3, those that have $a_1 = 5$ are greater than those that have $a_1 = 6, 7, 8, \ldots$ Therefore we can "build" q greedily, adding the largest unit fraction that keeps q less than $\frac{1}{4}$:
$\frac{1}{5}$ is the largest unit fraction less than $\frac{1}{4}$, hence $a_1 = 5$;
$\frac{1}{21}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5}$, hence $a_2 = 21$;
$\frac{1}{421}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5} - \frac{1}{21}$, hence $a_3 = 421$.

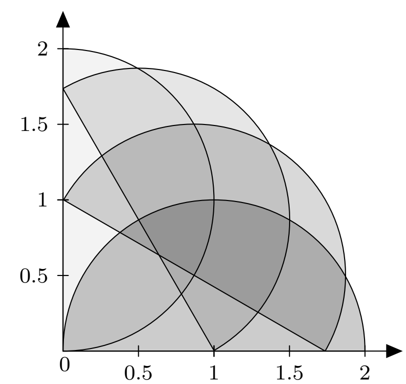
[1] http://en.wikipedia.org/wiki/Inscribed_angle_theorem

3. [**4**] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

Answer: $\boxed{39}$ Note that $1 = 0!$, $2 = 0! + 1!$, $3 = 0! + 2!$, and $4 = 0! + 1! + 2!$. These are the only numbers less than 6 that can be written as the sum of factorials. The only other factorials less than 240 are $3! = 6$, $4! = 24$, and $5! = 120$. So a positive integer less than or equal to 240 can only contain 3!, 4!, 5!, and/or one of 1, 2, 3, or 4 in its sum. If it contains any factorial larger than 5!, it will be larger than 240. So a sum less than or equal to 240 will will either include 3! or not (2 ways), 4! or not (2 ways), 5! or not (2 ways), and add an additional $0, 1, 2, 3$ or 4 (5 ways). This gives $2 \cdot 2 \cdot 2 \cdot 5 = 40$ integers less than 240. However, we want only positive integers, so we must not count 0. So there are 39 such positive integers.

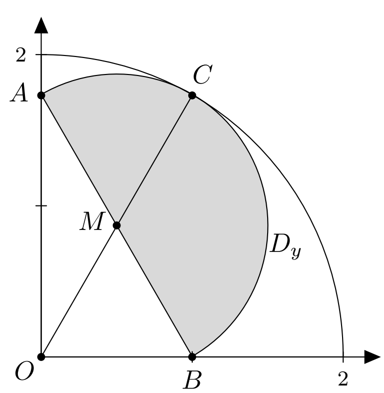
4. [**4**] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

Answer: $\boxed{\pi}$



From the picture above, we see that the union of the half-disks will be a quarter-circle with radius 2, and therefore area π. To prove that this is the case, we first prove that the boundary of every half-disk intersects the quarter-circle with radius 2, and then that the half-disk is internally tangent to the quarter-circle at that point. This is sufficient because it is clear from the diagram that we need not worry about covering the interior of the quarter-circle.

Let O be the origin. For a given half-disk D_y, label the vertex on the y-axis A and the vertex on the x-axis B. Let M be the midpoint of line segment $\overline{AB}$. Draw segment OM, and extend it until it intersects the curved boundary of D_y. Label the intersection point C. This construction is shown in the diagram below.



We first prove that C lies on the quarter-circle, centered at the origin, with radius 2. Since M is the midpoint of $\overline{AB}$, and A is on the y-axis, M is horizontally halfway between B and the y-axis. Since

O and B are on the x-axis (which is perpendicular to the y-axis), segments $\overline{OM}$ and $\overline{MB}$ have the same length. Since M is the midpoint of $\overline{AB}$, and $AB = 2$, $OM = 1$. Since D_y is a half-disk with radius 1, all points on its curved boundary are 1 away from its center, M. Then C is 2 away from the origin, and the quarter-circle consists of all points which are 2 away from the origin. Thus, C is an intersection of the half-disk D_y with the positive quarter-circle of radius 2.

It remains to show that the half-disk D_y is internally tangent to the quarter-circle. Since $\overline{OC}$ is a radius of the quarter-circle, it is perpendicular to the tangent of the quarter-circle at C. Since $\overline{MC}$ is a radius of the half-disk, it is perpendicular to the tangent of the half-disk at C. Then the tangents lines of the half-disk and the quarter-circle coincide, and the half-disk is tangent to the quarter-circle. It is obvious from the diagram that the half-disk lies at least partially inside of the quarter-circle, the half-disk D_y is internally tangent to the quarter-circle.

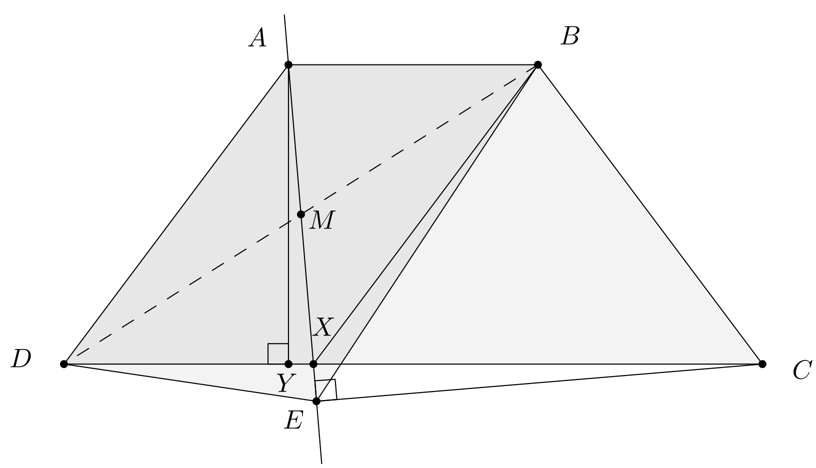
Then the union of the half-disks is be a quarter-circle with radius 2, and has area π. $\square$

5. [**5**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

Answer: $\boxed{1, -1, i, -i}$ Let k be a root of both polynomials. Multiplying the first polynomial by k and subtracting the second, we have $ak^4 - a = 0$, which means that k is either 1, -1, i, or $-i$. If $a = b = c = d = 1$, then -1, i, and $-i$ are roots of both polynomials. If $a = b = c = 1$ and $d = -3$, then 1 is a root of both polynomials. So k can be 1, -1, i, and $-i$.

6. [**5**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

Answer: $\boxed{\frac{216}{\sqrt{145}}}$



The locus of points E such that $[AED] = [AEB]$ forms a line, since area is a linear function of the coordinates of E; setting the areas equal gives a linear equation in the coordinates E[2]. Note that A and M, the midpoint of $\overline{DB}$, are on this line; A because both areas are 0, and M because the triangles share an altitude, and bases $\overline{DM}$ and $\overline{MB}$ are equal in length. Then $\overline{AM}$ is the set of points satisfying the area condition. The point E, then, is such that $\triangle AEC$ is a right angle (to make the distance minimal) and E lies on $\overline{AM}$.

Let X be the point of intersection of $\overline{AM}$ and $\overline{CD}$. Then $\triangle AMB \sim \triangle XMD$, and since $MD = BM$, they are in fact congruent. Thus $DX = AB = 10$, and $XC = 18$. Similarly, $BX = 15$, so $ABXD$ is a parallelogram. Let Y be the foot of the perpendicular from A to $\overline{DC}$, so that $DY = \frac{DC-AB}{2} = 9$. Then

[2] The notation $[A_1A_2 \dots A_n]$ means the area of polygon $A_1A_2 \dots A_n$.

$AY = \sqrt{AD^2 - DY^2} = \sqrt{225 - 81} = 12$. Then $YX = DX - DY = 1$ and $AX = \sqrt{AY^2 + YX^2} = \sqrt{144 + 1} = \sqrt{145}$. Since both $\triangle AXY$ and $\triangle CXE$ have a right angle, and $\angle EXC$ and $\angle YXA$ are congruent because they are vertical angles, $\triangle AXY \sim \triangle CXE$. Then $\frac{CE}{AY} = \frac{CX}{AX}$, so $CE = 12 \cdot \frac{18}{\sqrt{145}} = \frac{216}{\sqrt{145}}$.

7. [**5**] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

Answer: $\boxed{-90}$ We have $x^2 + y^2 + 2xy = 1$. Define $a = 2xy$ and $b = x^2 + y^2$ for convenience. Then $a + b = 1$ and $b - a = x^2 + y^2 - 2xy = (x - y)^2 = 2b - 1$ so that $x, y = \frac{\sqrt{2b-1} \pm 1}{2}$. Then

$$\begin{aligned}
x^{20} + y^{20} &= \left(\frac{\sqrt{2b-1}+1}{2}\right)^{20} + \left(\frac{\sqrt{2b-1}-1}{2}\right)^{20} \\
&= \frac{1}{2^{20}}[(\sqrt{2b-1}+1)^{20} + (\sqrt{2b-1}-1)^{20}] \\
&= \frac{2}{2^{20}}\left[(\sqrt{2b-1})^{20} + \binom{20}{2}(\sqrt{2b-1})^{18} + \binom{20}{4}(\sqrt{2b-1})^{16} + \dots\right] \\
&= \frac{2}{2^{20}}\left[(2b-1)^{10} + \binom{20}{2}(2b-1)^{9} + \binom{20}{4}(2b-1)^{8} + \dots\right] \\
&= 20
\end{aligned}$$

We want to find the sum of distinct roots of the above polynomial in b; we first prove that the original polynomial is square-free. The conditions $x+y = 1$ and $x^{20}+y^{20} = 20$ imply that $x^{20}+(1-x)^{20}-20 = 0$; let $p(x) = x^{20} + (1 - x)^{20} - 20$. p is square-free if and only if $GCD(p, p') = c$ for some constant c:

$$\begin{aligned}
GCD(p, p') &= GCD(x^{20} + (1 - x)^{20} - 20, 20(x^{19} - (1 - x)^{19})) \\
&= GCD(x^{20} - x(1 - x)^{19} + (1 - x)^{19} - 20, 20(x^{19} - (1 - x)^{19})) \\
&= GCD((1 - x)^{19} - 20, x^{19} - (1 - x)^{19}) \\
&= GCD((1 - x)^{19} - 20, x^{19} - 20)
\end{aligned}$$

The roots of $x^{19} - 20$ are $\sqrt[19]{20^k}\exp(\frac{2\pi ik}{19})$ for some $k = 0, 1, \dots, 18$; the roots of $(1 - x)^{19} - 20$ are $1 - \sqrt[19]{20^k}\exp(\frac{2\pi ik}{19})$ for some $k = 0, 1, \dots, 18$. If $x^{19} - 20$ and $(1 - x)^{19} - 20$ share a common root, then there exist integers m, n such that $\sqrt[19]{20^m}\exp(\frac{2\pi im}{19}) = 1 - \sqrt[19]{20^n}\exp(\frac{2\pi in}{19})$; since the imaginary parts of both sides must be the same, we have $m = n$ and $\sqrt[19]{20^m}\exp(\frac{2\pi im}{19}) = \frac{1}{2} \implies 20^m = \frac{1}{2^{19}}$, a contradiction. Thus we have proved that the polynomial in x has no double roots. Since for each b there exists a unique pair (x, y) (up to permutations) that satisfies $x^2 + y^2 = b$ and $(x + y)^2 = 1$, the polynomial in b has no double roots.

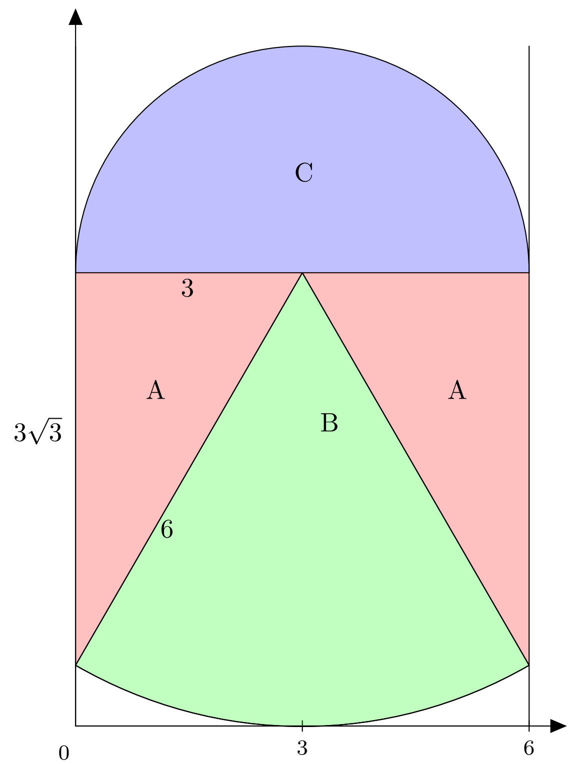
Let the coefficient of b^n in the above equation be $[b^n]$. By Vieta's Formulas, the sum of all possible values of $b = x^2 + y^2$ is equal to $-\frac{[b^9]}{[b^{10}]}$. $[b^{10}] = \frac{2}{2^{20}}\left(2^{10}\right)$ and $[b^9] = \frac{2}{2^{20}}\left(-\binom{10}{1}2^9 + \binom{20}{2}2^9\right)$; thus $-\frac{[b^9]}{[b^{10}]} = -\frac{\binom{10}{1}2^9 - \binom{20}{2}2^9}{2^{10}} = -90$.

8. [**6**] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y), it can move to $(x - 1, y - 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$, or $(x + 1, y + 1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

Answer: $\boxed{\binom{4020}{1005}^2}$ Note that each of the coordinates either increases or decreases the x and y-coordinates by 1. In order to reach 2010 after 4020 steps, each of the coordinates must be increased 3015 times and decreased 1005 times. A permutation of 3015 plusses and 1005 minuses for each of x and y uniquely corresponds to a path the ant could take to $(2010, 2010)$, because we can take ordered pairs from the two lists and match them up to a valid step the ant can take. So the number of ways the ant can end up at $(2010, 2010)$ after 4020 seconds is equal to the number of ways to arrange plusses and minuses for both x and y, or $(\binom{4020}{1005})^2$.

9. [**7**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

Answer: $\boxed{9\sqrt{3} + \frac{21\pi}{2}}$



If you draw concentric circles around the destination point, the condition is equivalent to the restriction that you must always go inwards towards the destination. In the diagram above, the regions through which you might pass are shaded.

We find the areas of regions A, B, and C separately, and add them up (doubling the area of region A, because there are two of them).

The hypotenuse of triangle A is of length 6, and the base is of length 3, so it is a $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ triangle (30-60-90 triangle) with area $\frac{9\sqrt{3}}{2}$. Then the total area of the regions labeled A is $9\sqrt{3}$.

Since the angle of triangle A nearest the center of the circle (the destination point) is $\frac{\pi}{3}$, sector B has central angle $\frac{\pi}{3}$. Then the area of sector B is $\frac{1}{2}r^2\theta = \frac{1}{2} \cdot 36 \cdot \frac{\pi}{3} = 6\pi$.

Region C is a half-disc of radius 3, so its area is $\frac{9\pi}{2}$.

Thus, the total area is $9\sqrt{3} + \frac{21\pi}{2}$.

10. [**8**] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?

Answer: $\boxed{49}$ First, we show that 50 is too big. Assume for sake of contradiction that a labeling with at least 50 distinct integers exists. By the Pigeonhole Principle, there must be at least one row, say the first row, with at least 4 distinct integers in it; in this case, that is exactly 4, since that is the maximum number of distinct integers in one row. Then, in the remaining 15 rows there must be at least 46 distinct integers (these 46 will also be distinct from the 4 in the first row). Using Pigeonhole again, there will be another row, say the second row, with 4 distinct integers in it. Call the set of integers in the first and second rows S. Because the 4 distinct integers in the second row are distinct from the 4 in the first row, there are 8 distinct values in the first two rows, so $|S| = 8$. Now consider the subcolumns containing the cells in rows 3 to 16. In each subcolumn, there are at most 2 values not in S, because there are already two distinct values in that column from the cells in the first two rows. So, the maximum number of distinct values in the table is $16 \cdot 2 + 8 = 40$, a contradiction. So a valid labeling must have fewer than 50 distinct integers. Below, we show by example that 49 is attainable.

1	17	33	-	-	-	-	-	-	-	-	-	-	-	-	-
-	2	18	34	-	-	-	-	-	-	-	-	-	-	-	-
-	-	3	19	35	-	-	-	-	-	-	-	-	-	-	-
-	-	-	4	20	36	-	-	-	-	-	-	-	-	-	-
-	-	-	-	5	21	37	-	-	-	-	-	-	-	-	-
-	-	-	-	-	6	22	38	-	-	-	-	-	-	-	-
-	-	-	-	-	-	7	23	39	-	-	-	-	-	-	-
-	-	-	-	-	-	-	8	24	40	-	-	-	-	-	-
-	-	-	-	-	-	-	-	9	25	41	-	-	-	-	-
-	-	-	-	-	-	-	-	-	10	26	42	-	-	-	-
-	-	-	-	-	-	-	-	-	-	11	27	43	-	-	-
-	-	-	-	-	-	-	-	-	-	-	12	28	44	-	-
-	-	-	-	-	-	-	-	-	-	-	-	13	29	45	-
-	-	-	-	-	-	-	-	-	-	-	-	-	14	30	46
47	-	-	-	-	-	-	-	-	-	-	-	-	-	15	31
32	48	-	-	-	-	-	-	-	-	-	-	-	-	-	16

Cells that do not contain a number are colored with color 49.