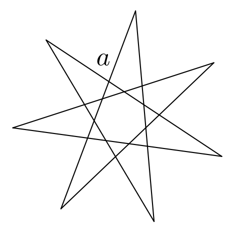
13th Annual Harvard-MIT Mathematics Tournament

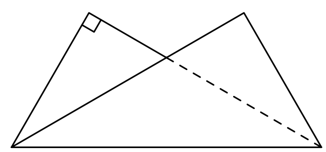
Saturday 20 February 2010

Geometry Subject Test

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



2. [**3**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



3. [**4**] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

4. [**4**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

5. [**4**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

6. [**5**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

7. [**6**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

8. [**6**] Let O be the point $(0, 0)$. Let A, B, C be three points in the plane such that $AO = 15$, $BO = 15$, and $CO = 7$, and such that the area of triangle ABC is maximal. What is the length of the shortest side of ABC?

9. [**7**] Let $ABCD$ be a quadrilateral with an inscribed circle centered at I. Let CI intersect AB at E. If $\angle IDE = 35^\circ$, $\angle ABC = 70^\circ$, and $\angle BCD = 60^\circ$, then what are all possible measures of $\angle CDA$?

10. [**8**] Circles ω_1 and ω_2 intersect at points A and B. Segment PQ is tangent to ω_1 at P and to ω_2 at Q, and A is closer to PQ than B. Point X is on ω_1 such that $PX \parallel QB$, and point Y is on ω_2 such that $QY \parallel PB$. Given that $\angle APQ = 30^\circ$ and $\angle PQA = 15^\circ$, find the ratio AX/AY.