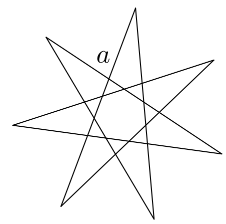
13th Annual Harvard-MIT Mathematics Tournament

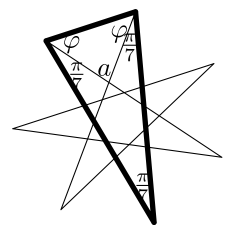
Saturday 20 February 2010

Geometry Subject Test

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



Answer: $\boxed{\frac{3\pi}{7}}$ The measure of the interior angle of a point of the star is $\frac{\pi}{7}$ because it is an inscribed angle on the circumcircle which intercepts a seventh of the circle.[1]



Consider the triangle shown above in bold. Because the sum of the angles in any triangle is π,

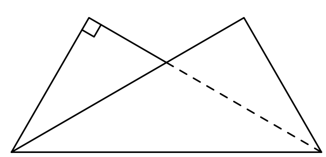
$$2\varphi + 3\left(\frac{\pi}{7}\right) = \pi = 2\varphi + a$$

Canceling the 2φ on the right-hand side and on the left-hand side, we obtain

$$a = \frac{3\pi}{7}.$$

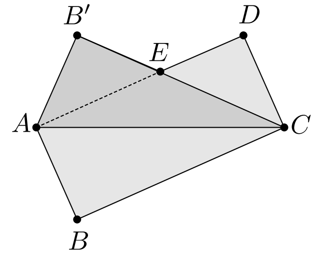
□

2. [**3**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



Answer: $\boxed{\sqrt{5}}$

[1] http://en.wikipedia.org/wiki/Inscribed_angle_theorem



Given a polygon $P_1P_2\cdots P_k$, let $[P_1P_2\cdots P_k]$ denote its area. Let $ABCD$ be the rectangle. Suppose we fold B across $\overline{AC}$, and let E be the intersection of $\overline{AD}$ and $\overline{B'C}$. Then we end up with the pentagon $ACDEB'$, depicted above. Let's suppose, without loss of generality, that $ABCD$ has area 1. Then $\triangle AEC$ must have area $\frac{3}{10}$, since

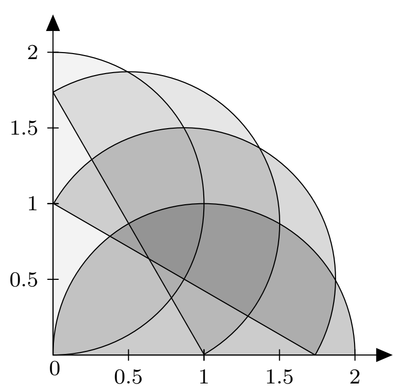
$$\begin{aligned}[ABCD] &= [ABC]+[ACD]\\ &= [AB'C]+[ACD]\\ &= [AB'E]+2[AEC]+[EDC]\\ &= [ACDEB']+[AEC]\\ &= \frac{7}{10}[ABCD]+[AEC],\end{aligned}$$

That is, $[AEC]=\frac{3}{10}[ABCD]=\frac{3}{10}$.

Since $\triangle ECD$ is congruent to $\triangle EAB'$, both triangles have area $\frac{1}{5}$. Note that $\triangle AB'C$, $\triangle ABC$, and $\triangle CDA$ are all congruent, and all have area $\frac{1}{2}$. Since $\triangle AEC$ and $\triangle EDC$ share altitude $\overline{DC}$, $\frac{DE}{EA}=\frac{[DEC]}{[AEC]}=\frac{2}{3}$. Because $\triangle CAE$ is isosceles, $CE=EA$. Let $AE=3x$. The $CE=3x$, $DE=2x$, and $CD=x\sqrt{9-4}=x\sqrt{5}$. Then $\frac{AD}{DC}=\frac{AE+ED}{DC}=\frac{3+2}{\sqrt{5}}=\sqrt{5}$.

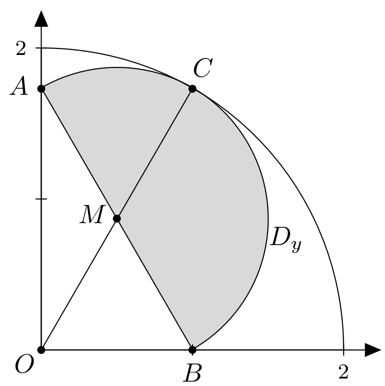
3. [**4**] For $0\le y\le 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0,y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0\le y\le 2$.

Answer: $\boxed{\pi}$



From the picture above, we see that the union of the half-disks will be a quarter-circle with radius 2, and therefore area π. To prove that this is the case, we first prove that the boundary of every half-disk intersects the quarter-circle with radius 2, and then that the half-disk is internally tangent to the quarter-circle at that point. This is sufficient because it is clear from the diagram that we need not worry about covering the interior of the quarter-circle.

Let O be the origin. For a given half-disk D_y, label the vertex on the y-axis A and the vertex on the x-axis B. Let M be the midpoint of line segment $\overline{AB}$. Draw segment OM, and extend it until it intersects the curved boundary of D_y. Label the intersection point C. This construction is shown in the diagram below.



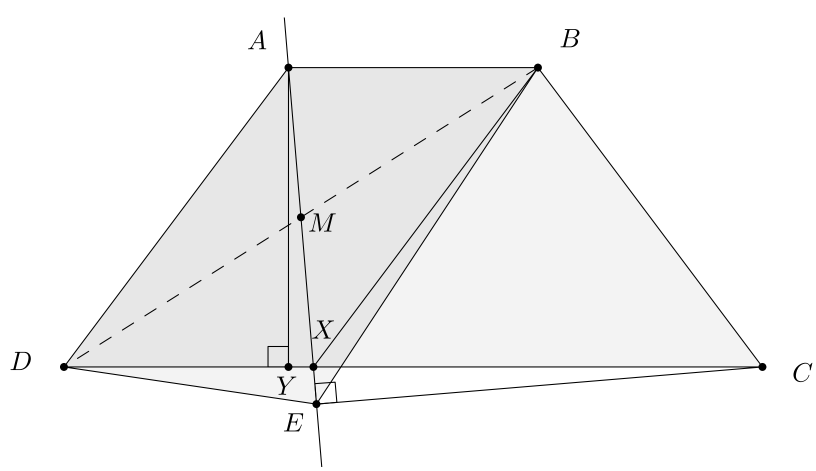
We first prove that C lies on the quarter-circle, centered at the origin, with radius 2. Since M is the midpoint of $\overline{AB}$, and A is on the y-axis, M is horizontally halfway between B and the y-axis. Since O and B are on the x-axis (which is perpendicular to the y-axis), segments $\overline{OM}$ and $\overline{MB}$ have the same length. Since M is the midpoint of $\overline{AB}$, and $AB = 2$, $OM = 1$. Since D_y is a half-disk with radius 1, all points on its curved boundary are 1 away from its center, M. Then C is 2 away from the origin, and the quarter-circle consists of all points which are 2 away from the origin. Thus, C is an intersection of the half-disk D_y with the positive quarter-circle of radius 2.

It remains to show that the half-disk D_y is internally tangent to the quarter-circle. Since $\overline{OC}$ is a radius of the quarter-circle, it is perpendicular to the tangent of the quarter-circle at C. Since $\overline{MC}$ is a radius of the half-disk, it is perpendicular to the tangent of the half-disk at C. Then the tangents lines of the half-disk and the quarter-circle coincide, and the half-disk is tangent to the quarter-circle. It is obvious from the diagram that the half-disk lies at least partially inside of the quarter-circle, the half-disk D_y is internally tangent to the quarter-circle.

Then the union of the half-disks is be a quarter-circle with radius 2, and has area π. □

4. [**4**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

Answer: $\boxed{\frac{216}{\sqrt{145}}}$



The locus of points E such that $[AED] = [AEB]$ forms a line, since area is a linear function of the coordinates of E; setting the areas equal gives a linear equation in the coordinates E^2. Note that A and M, the midpoint of $\overline{DB}$, are on this line; A because both areas are 0, and M because the triangles share an altitude, and bases $\overline{DM}$ and $\overline{MB}$ are equal in length. Then $\overline{AM}$ is the set of points satisfying the area condition. The point E, then, is such that $\triangle AEC$ is a right angle (to make the distance minimal) and E lies on $\overline{AM}$.

Let X be the point of intersection of $\overline{AM}$ and $\overline{CD}$. Then $\triangle AMB \sim \triangle XMD$, and since $MD = BM$, they are in fact congruent. Thus $DX = AB = 10$, and $XC = 18$. Similarly, $BX = 15$, so $ABXD$ is a parallelogram. Let Y be the foot of the perpendicular from A to $\overline{DC}$, so that $DY = \frac{DC-AB}{2} = 9$. Then $AY = \sqrt{AD^2 - DY^2} = \sqrt{225 - 81} = 12$. Then $YX = DX - DY = 1$ and $AX = \sqrt{AY^2 + YX^2} = \sqrt{144 + 1} = \sqrt{145}$. Since both $\triangle AXY$ and $\triangle CXE$ have a right angle, and $\angle EXC$ and $\angle YXA$ are congruent because they are vertical angles, $\triangle AXY \sim \triangle CXE$. Then $\frac{CE}{AY} = \frac{CX}{AX}$, so $CE = 12 \cdot \frac{18}{\sqrt{145}} = \frac{216}{\sqrt{145}}$.

5. [**4**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

Answer: $\boxed{2}$ The answer is 2 for any number of dimensions. We prove this by induction on the dimension.

Note that 1-dimensional spheres are pairs of points, and 2-dimensional spheres are circles.

Base case, $d = 2$: The intersection of two circles is either a circle (if the original circles are identical, and in the same place), a pair of points, a single point (if the circles are tangent), or the empty set. Thus, in dimension 2, the largest finite number of intersection points is 2, because the number of pairwise intersection points is 0, 1, or 2 for distinct circles.

We now prove that the intersection of two k-dimensional spheres is either the empty set, a $(k-1)$-dimensional sphere, a k-dimensional sphere (which only occurs if the original spheres are identical and coincident). Consider two spheres in k-dimensional space, and impose a coordinate system such that the centers of the two spheres lie on one coordinate axis. Then the equations for the two spheres become identical in all but one coordinate:

$$\begin{aligned}(x_1 - a_1)^2 + x_2^2 + \ldots + x_k^2 &= r_1^2 \\ (x_1 - a_2)^2 + x_2^2 + \ldots + x_k^2 &= r_2^2\end{aligned}$$

If $a_1 = a_2$, the spheres are concentric, and so they are either nonintersecting or coincident, intersecting in a k-dimensional sphere. If $a_1 \neq a_2$, then subtracting the equations and solving for x_1 yields $x_1 = \frac{r_1^2 - a_1^2 - r_2^2 + a_2^2}{2(a_2 - a_2)}$. Plugging this in to either equation above yields a single equation equation that describes a $(k-1)$-dimensional sphere.

Assume we are in dimension d, and suppose for induction that for all k less than d, any two distinct k-dimensional spheres intersecting in a finite number of points intersect in at most two points. Suppose we have a collection of d-dimensional spheres $s_1, s_2, \ldots, s_m$. Without loss of generality, suppose the s_i are distinct. Let t_i be the intersection of s_i and s_{i+1} for $1 \leq i < m$. If any t_i are the empty set, then the intersection of the t_i is empty. None of the t_i is a d-dimensional sphere because the s_i are distinct. Thus each of $t_1, t_2, \ldots, t_{m-1}$ is a $(d-1)$-dimensional sphere, and the intersection of all of them is the same as the intersection of the d-dimensional spheres. We can then apply the inductive hypothesis to find that $t_1, \ldots, t_{m-1}$ intersect in at most two points. Thus, by induction, a set of spheres in any dimension which intersect at only finitely many points intersect at at most two points.

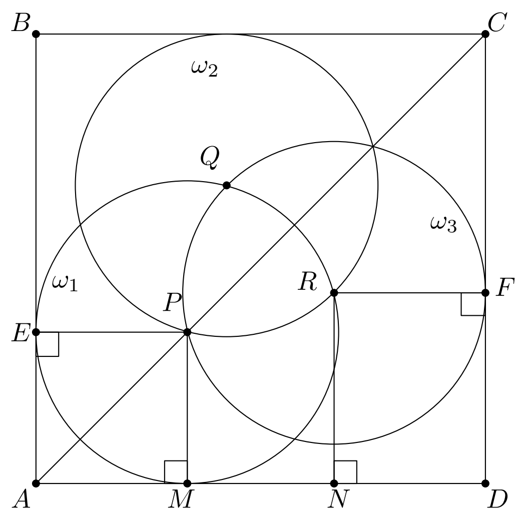
We now exhibit a set of 2^{2009} 2010-dimensional spheres, and prove that their intersection contains exactly two points. Take the spheres with radii $\sqrt{2013}$ and centers $(0, \pm 1, \pm 1, \ldots, \pm 1)$, where the sign of each coordinate is independent from the sign of every other coordinate. Because of our choice of radius, all these spheres pass through the points $(\pm 2, 0, 0, \ldots 0)$. Then the intersection is the set of

[2]The notation $[A_1 A_2 \ldots A_n]$ means the area of polygon $A_1 A_2 \ldots A_n$.

points $(x_1, x_2, \ldots, x_{2010})$ which satisfy the equations $x_1^2 + (x_2 \pm 1)^2 + \cdots + (x_{2010} \pm 1)^2 = 2013$. The only solutions to these equations are the points $(\pm 2, 0, 0, \ldots, 0)$ (since $(x_i + 1)^2$ must be the same as $(x_i - 1)^2$ for all $i > 1$, because we may hold all but one of the $\pm$ choices constant, and change the remaining one). $\square$

6. [**5**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

Answer: $\boxed{\frac{\sqrt{6}+\sqrt{2}+8}{4}}$



By the Pigeonhole Principle, two of the sides must be tangent to the same circle, say ω_1. Since S surrounds the circles, these two sides must be adjacent, so we can let A denote the common vertex of the two sides tangent to ω_1. Let B, C, and D be the other vertices of S in clockwise order, and let P, Q, and R be the centers of ω_1, ω_2, and ω_3 respectively, and suppose WLOG that they are also in clockwise order. Then AC passes through the center of ω_1, and by symmetry (since $AB = AD$) it must also pass through the other intersection point of ω_2 and ω_3. That is, AC is the radical axis of ω_2 and ω_3.

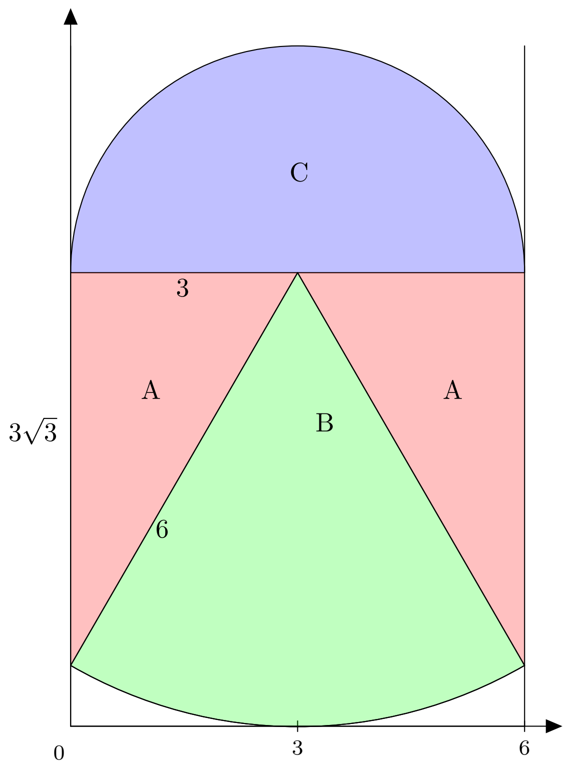
Now, let M and N be the feet of the perpendiculars from P and R, respectively, to side AD. Let E and F be the feet of the perpendiculars from P to AB and from R to DC, respectively. Then $PEAM$ and $NRFD$ are rectangles, and PE and RF are radii of ω_1 and ω_2 respectively. Thus $AM = EP = 1$ and $ND = RF = 1$. Finally, we have

$$\begin{aligned} MN &= PR \cdot \cos(180^\circ - \angle EPR) \\ &= \cos(180^\circ - EPQ - RPQ) \\ &= -\cos\left((270^\circ - 60^\circ)/2 + 60^\circ\right) \\ &= -\cos(165^\circ) \\ &= \cos(15^\circ) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

Thus $AD = AM + MN + ND = 1 + \frac{\sqrt{6}+\sqrt{2}}{4} + 1 = \frac{\sqrt{6}+\sqrt{2}+8}{4}$ as claimed. $\square$

7. [**6**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

   Answer: $\boxed{9\sqrt{3} + \frac{21\pi}{2}}$



If you draw concentric circles around the destination point, the condition is equivalent to the restriction that you must always go inwards towards the destination. In the diagram above, the regions through which you might pass are shaded.

We find the areas of regions A, B, and C separately, and add them up (doubling the area of region A, because there are two of them).

The hypotenuse of triangle A is of length 6, and the base is of length 3, so it is a $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ triangle (30-60-90 triangle) with area $\frac{9\sqrt{3}}{2}$. Then the total area of the regions labeled A is $9\sqrt{3}$.

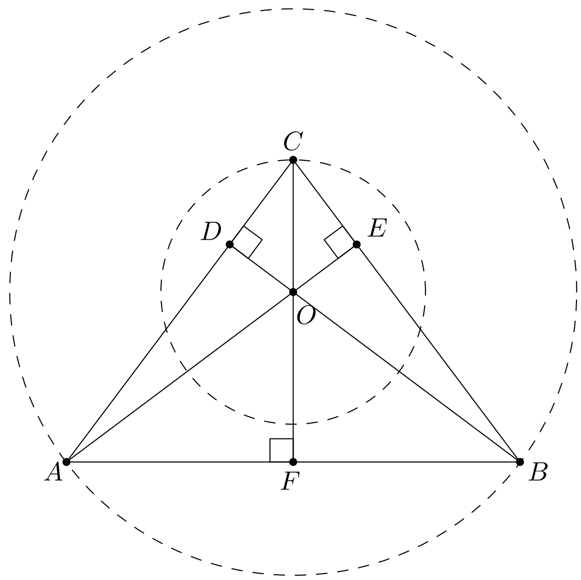
Since the angle of triangle A nearest the center of the circle (the destination point) is $\frac{\pi}{3}$, sector B has central angle $\frac{\pi}{3}$. Then the area of sector B is $\frac{1}{2}r^2\theta = \frac{1}{2} \cdot 36 \cdot \frac{\pi}{3} = 6\pi$.

Region C is a half-disc of radius 3, so its area is $\frac{9\pi}{2}$.

Thus, the total area is $9\sqrt{3} + \frac{21\pi}{2}$.

8. [**6**] Let O be the point $(0, 0)$. Let A, B, C be three points in the plane such that $AO = 15$, $BO = 15$, and $CO = 7$, and such that the area of triangle ABC is maximal. What is the length of the shortest side of ABC?

   Answer: $\boxed{20}$ We claim that O should be the orthocenter of the triangle ABC. If O is not on an altitude of $\triangle ABC$, suppose (without loss of generality) that $\overline{AO}$ is not perpendicular to $\overline{BC}$. We can rotate A around O, leaving B and C fixed, to make $\overline{AO}$ perpendicular to $\overline{BC}$, which strictly increases the area. Therefore, if $[ABC]$ is maximal then $\triangle ABC$ is an isosceles triangle with orthocenter O and base $\overline{AB}$.

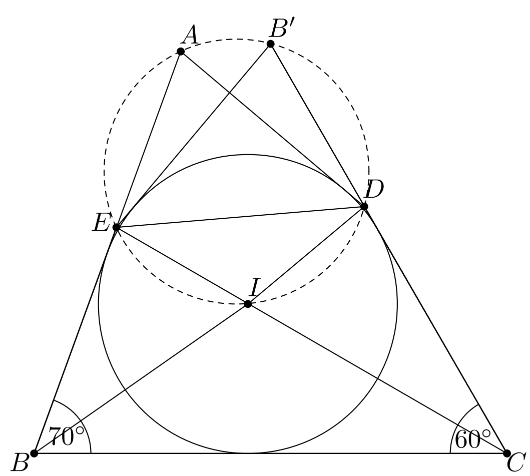


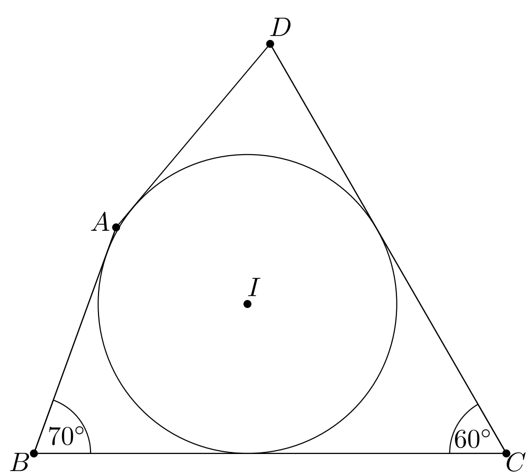
Let F be the foot of the perpendicular from C to $\overline{AB}$. Since $\angle FOA$ and $\angle COE$ are vertical, $\angle FAO = \angle OCE$. Then $\triangle FAO$ is similar to $\triangle FCB$, so we have $\frac{AF}{OF} = \frac{CF}{BF} = \frac{OF+7}{AF}$, so $AF^2 = OF^2 + 7 \cdot OF$. Since $AF^2 = 225 - OF^2$, $2 \cdot OF^2 + 7 \cdot OF - 225 = 0$, so $OF = 9$. Then $AF = 12$, so $AB = 24$ and $BC = 20$. Thus, the length of the shortest side of $\triangle ABC$ is 20.

9. [**7**] Let $ABCD$ be a quadrilateral with an inscribed circle centered at I. Let CI intersect AB at E. If $\angle IDE = 35°$, $\angle ABC = 70°$, and $\angle BCD = 60°$, then what are all possible measures of $\angle CDA$?

Answer: $\boxed{70° \text{ and } 160°}$

Arbitrarily defining B and C determines I and E up to reflections across BC. D lies on both the circle determined by $\angle EDI = 35°$ and the line through C tangent to the circle (and on the opposite side of B); since the intersection of a line and a circle has at most two points, there are only two cases for $ABCD$. The diagram below on the left shows the construction made in this solution, containing both cases. The diagram below on the right shows only the degenerate case.





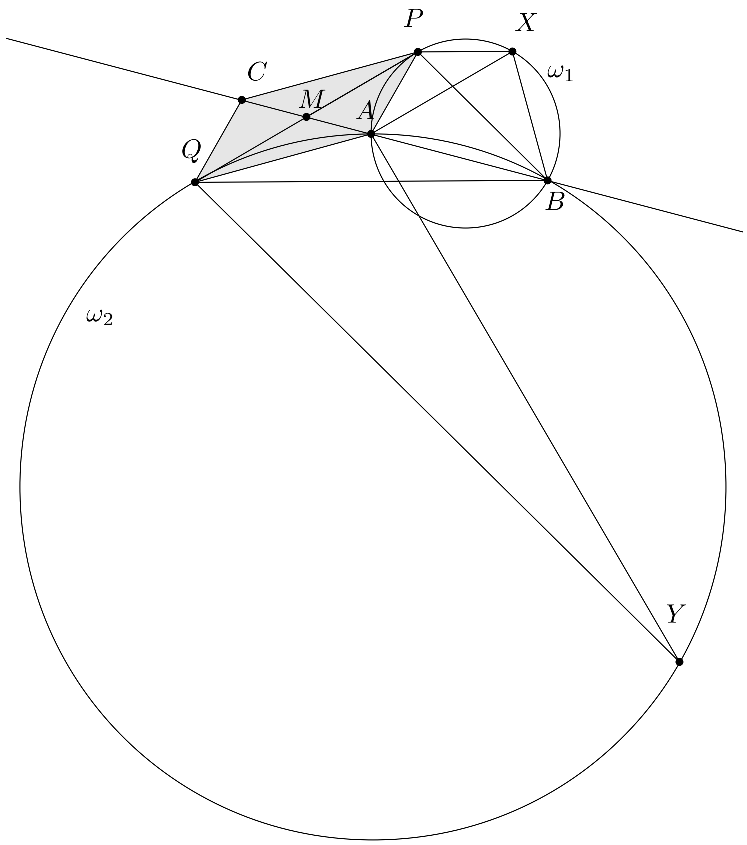
Reflect B across EC to B'; then $CB = CB'$. Since BA and BC are tangent to the circle centered at I, IB is the angle bisector of $\angle ABC$. Then $\angle IBE = \angle IB'E = 35°$. If $B' = D$, then $\angle ADC = \angle EB'C = 70°$. Otherwise, since $\angle IB'E = 35° = \angle IDE$ (given), $EB'DI$ is a cyclic quadrilateral. Then $\angle IED = \angle IB'D = 35°$ and $\angle BCI = \angle ECD = 30°$, so $\triangle CED \sim \triangle CBI$.

Since $\angle CID$ is exterior to $\triangle DIE$, $\angle CID = \angle IDE + \angle DEI = 70°$. Then $\triangle CDI \sim \triangle CEB$. Because $EB'DI$ is cyclic, $\angle IDC = \angle IEB' = \angle IEB = 180° - 70° - 30° = 80°$. Then $\angle ADC = 2\angle IDC = 160°$.

Thus, the two possible measures are $70°$ and $160°$.

10. [**8**] Circles ω_1 and ω_2 intersect at points A and B. Segment PQ is tangent to ω_1 at P and to ω_2 at Q, and A is closer to PQ than B. Point X is on ω_1 such that $PX \parallel QB$, and point Y is on ω_2 such that $QY \parallel PB$. Given that $\angle APQ = 30°$ and $\angle PQA = 15°$, find the ratio AX/AY.

Answer: $\boxed{2 - \sqrt{3}}$



Let C be the fourth vertex of parallelogram $APCQ$. The midpoint M of $\overline{PQ}$ is the intersection of the diagonals of this parallelogram. Because M has equal power[3] with respect to the two circles ω_1 and ω_2, it lies on $\overleftrightarrow{AB}$, the circles' radical axis[4]. Therefore, C lies on $\overleftrightarrow{AB}$ as well.

Using a series of parallel lines and inscribed arcs, we have:

$$\angle APC = \angle APQ + \angle CPQ = \angle APQ + \angle PQA = \angle ABP + \angle QBA = \angle PBQ = \angle XPB,$$

where the last equality follows from the fact that $PX \parallel QB$.

[3]`http://en.wikipedia.org/wiki/Power_of_a_point`
[4]`http://en.wikipedia.org/wiki/Radical_axis`

We also know that $\angle BXP = 180^\circ - \angle PAB = \angle CAP$, so triangles BXP and CAP are similar. By the spiral similarity theorem[5], triangles BPC and XPA are similar, too.

By analogous reasoning, triangles BQC and YQA are similar. Then we have:

$$\frac{AX}{AY} = \frac{AX/BC}{AY/BC} = \frac{AP/CP}{AQ/CQ} = \frac{AP^2}{AQ^2}$$

where the last inequality holds because $APCQ$ is a parallelogram. Using the Law of Sines, the last expression equals $\frac{\sin^2 15^\circ}{\sin^2 30^\circ} = 2 - \sqrt{3}$.

[5]This theorem states that if $\triangle PAB$ and $\triangle PXY$ are similar and oriented the same way, then $\triangle PAX$ and $\triangle PBY$ are similar too. It is true because the first similarity implies that $AP/BP = XP/YP$ and $\angle APB = \angle XPY$, which proves the second similarity.