13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Guts Round

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1. [**4**] If $A = 10^9 - 987654321$ and $B = \frac{123456789+1}{10}$, what is the value of $\sqrt{AB}$?

2. [**4**] Suppose that x, y, and z are non-negative real numbers such that $x + y + z = 1$. What is the maximum possible value of $x + y^2 + z^3$?

3. [**4**] In a group of people, there are 13 who like apples, 9 who like blueberries, 15 who like cantaloupe, and 6 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes blueberries also likes exactly one of apples and cantaloupe. Each person who likes cantaloupe also likes exactly one of blueberries and dates. Find the minimum possible number of people in the group.

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4. [**5**] To set up for a Fourth of July party, David is making a string of red, white, and blue balloons. He places them according to the following rules:

   - No red balloon is adjacent to another red balloon.
   - White balloons appear in groups of exactly two, and groups of white balloons are separated by at least two non-white balloons.
   - Blue balloons appear in groups of exactly three, and groups of blue balloons are separated by at least three non-blue balloons.

   If David uses over 600 balloons, determine the smallest number of red balloons that he can use.

5. [**5**] You have a length of string and 7 beads in the 7 colors of the rainbow. You place the beads on the string as follows – you randomly pick a bead that you haven't used yet, then randomly add it to either the left end or the right end of the string. What is the probability that, at the end, the colors of the beads are the colors of the rainbow in order? (The string cannot be flipped, so the red bead must appear on the left side and the violet bead on the right side.)

6. [**5**] How many different numbers are obtainable from five 5s by first concatenating some of the 5s, then multiplying them together? For example, we could do $5 \cdot 55 \cdot 55$, $555 \cdot 55$, or 55555, but not $5 \cdot 5$ or 2525.

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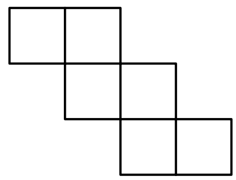
7. [**6**] What are the last 8 digits of

$$11 \times 101 \times 1001 \times 10001 \times 100001 \times 1000001 \times 111?$$

8. [**6**] Each square in the following hexomino has side length 1. Find the minimum area of any rectangle that contains the entire hexomino.



9. [**6**] Indecisive Andy starts out at the midpoint of the 1-unit-long segment $\overline{HT}$. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of $\overline{HT}$?

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10. [**7**] Let ABC be a triangle with $AB = 8$, $BC = 15$, and $AC = 17$. Point X is chosen at random on line segment AB. Point Y is chosen at random on line segment BC. Point Z is chosen at random on line segment CA. What is the expected area of triangle XYZ?

11. [**7**] From the point (x, y), a *legal move* is a move to $(\frac{x}{3} + u, \frac{y}{3} + v)$, where u and v are real numbers such that $u^2 + v^2 \leq 1$. What is the area of the set of points that can be reached from $(0, 0)$ in a finite number of legal moves?

12. [**7**] How many different collections of 9 letters are there? A letter can appear multiple times in a collection. Two collections are equal if each letter appears the same number of times in both collections.

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13. [**8**] A triangle in the xy-plane is such that when projected onto the x-axis, y-axis, and the line $y = x$, the results are line segments whose endpoints are $(1, 0)$ and $(5, 0)$, $(0, 8)$ and $(0, 13)$, and $(5, 5)$ and $(7.5, 7.5)$, respectively. What is the triangle's area?

14. [**8**] In how many ways can you fill a 3×3 table with the numbers 1 through 9 (each used once) such that all pairs of adjacent numbers (sharing one side) are relatively prime?

15. [**8**] Pick a random integer between 0 and 4095, inclusive. Write it in base 2 (without any leading zeroes). What is the expected number of consecutive digits that are not the same (that is, the expected number of occurrences of either 01 or 10 in the base 2 representation)?

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16. [**9**] Jessica has three marbles colored red, green, and blue. She randomly selects a non-empty subset of them (such that each subset is equally likely) and puts them in a bag. You then draw three marbles from the bag with replacement. The colors you see are red, blue, red. What is the probability that the only marbles in the bag are red and blue?

17. [**9**] An ant starts at the origin, facing in the positive x-direction. Each second, it moves 1 unit forward, then turns counterclockwise by $\sin^{-1}(\frac{3}{5})$ degrees. What is the least upper bound on the distance between the ant and the origin? (The least upper bound is the smallest real number r that is at least as big as every distance that the ant ever is from the origin.)

18. [**9**] Find two lines of symmetry of the graph of the function $y = x + \frac{1}{x}$. Express your answer as two equations of the form $y = ax + b$.

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19. [**10**] A 5-dimensional ant starts at one vertex of a 5-dimensional hypercube of side length 1. A *move* is when the ant travels from one vertex to another vertex at a distance of $\sqrt{2}$ away. How many ways can the ant make 5 moves and end up on the same vertex it started at?

20. [**10**] Find the volume of the set of points (x, y, z) satisfying

$$\begin{aligned} x, y, z &\geq 0 \\ x + y &\leq 1 \\ y + z &\leq 1 \\ z + x &\leq 1 \end{aligned}$$

21. [**10**] Let $\triangle ABC$ be a scalene triangle. Let h_a be the locus of points P such that $|PB - PC| = |AB - AC|$. Let h_b be the locus of points P such that $|PC - PA| = |BC - BA|$. Let h_c be the locus of points P such that $|PA - PB| = |CA - CB|$. In how many points do all of h_a, h_b, and h_c concur?

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22. [**12**] You are the general of an army. You and the opposing general both have an equal number of troops to distribute among three battlefields. Whoever has more troops on a battlefield always wins (you win ties). An *order* is an ordered triple of non-negative real numbers (x, y, z) such that $x + y + z = 1$, and corresponds to sending a fraction x of the troops to the first field, y to the second, and z to the third. Suppose that you give the order $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and that the other general issues an order chosen uniformly at random from all possible orders. What is the probability that you win two out of the three battles?

23. [**12**] In the country of Francisca, there are 2010 cities, some of which are connected by roads. Between any two cities, there is a unique path which runs along the roads and which does not pass through any city twice. What is the maximum possible number of cities in Francisca which have at least 3 roads running out of them?

24. [**12**] Define a sequence of polynomials as follows: let $a_1 = 3x^2 - x$, let $a_2 = 3x^2 - 7x + 3$, and for $n \geq 1$, let $a_{n+2} = \frac{5}{2}a_{n+1} - a_n$. As n tends to infinity, what is the limit of the sum of the roots of a_n?

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25. [**15**] How many functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ have the property that $f(\{1,2,3\})$ and $f(f(\{1,2,3\}))$ are disjoint?

26. [**15**] Express the following in closed form, as a function of x:

$$\sin^2(x)+\sin^2(2x)\cos^2(x)+\sin^2(4x)\cos^2(2x)\cos^2(x)+\cdots+\sin^2(2^{2010}x)\cos^2(2^{2009}x)\cdots\cos^2(2x)\cos^2(x).$$

27. [**15**] Suppose that there are real numbers $a, b, c \geq 1$ and that there are positive reals x, y, z such that

$$\begin{aligned} a^x + b^y + c^z &= 4 \\ xa^x + yb^y + zc^z &= 6 \\ x^2a^x + y^2b^y + z^2c^z &= 9. \end{aligned}$$

What is the maximum possible value of c?

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28. [**18**] Danielle Bellatrix Robinson is organizing a poker tournament with 9 people. The tournament will have 4 rounds, and in each round the 9 players are split into 3 groups of 3. During the tournament, each player plays every other player exactly once. How many different ways can Danielle divide the 9 people into three groups in each round to satisfy these requirements?

29. [**18**] Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

30. [**18**] A monomial term $x_{i_1}x_{i_2}\ldots x_{i_k}$ in the variables $x_1, x_2, \ldots x_8$ is *square-free* if $i_1, i_2, \ldots i_k$ are distinct. (A constant term such as 1 is considered square-free.) What is the sum of the coefficients of the square-free terms in the following product?

$$\prod_{1\leq i<j\leq 8}(1+x_ix_j)$$

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31. [**21**] In the Democratic Republic of Irun, 5 people are voting in an election among 5 candidates. If each person votes for a single candidate at random, what is the expected number of candidates that will be voted for?

32. [**21**] There are 101 people participating in a Secret Santa gift exchange. As usual each person is randomly assigned another person for whom (s)he has to get a gift, such that each person gives and receives exactly one gift and no one gives a gift to themself. What is the probability that the first person neither gives gifts to or receives gifts from the second or third person? Express your answer as a decimal rounded to five decimal places.

33. [**21**] Let $a_1 = 3$, and for $n > 1$, let a_n be the largest real number such that

$$4\left(a_{n-1}^2 + a_n^2\right) = 10a_{n-1}a_n - 9.$$

What is the largest positive integer less than a_8?

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Note: Problems 34 and 36 are estimation problems. You will receive a number of points (between 0 and 25) based on how close your answer is to the correct answer. Throughout, we will let C denote the correct answer and A denote your answer. If a described scoring method would ever assign a negative number of points, you will receive zero points instead.

34. [**25**] 3000 people each go into one of three rooms randomly. What is the most likely value for the maximum number of people in any of the rooms? Your score for this problem will be 0 if you write down a number less than or equal to 1000. Otherwise, it will be $25 - 27\frac{|A-C|}{\min(A,C)-1000}$.

35. [**25**] Call an positive integer *almost-square* if it can be written as $a \cdot b$, where a and b are integers and $a \leq b \leq \frac{4}{3}a$. How many almost-square positive integers are less than or equal to 1000000? Your score will be equal to $25 - 65\frac{|A-C|}{\min(A,C)}$.

36. [**25**] Consider an infinite grid of unit squares. An n-*omino* is a subset of n squares that is connected. Below are depicted examples of 8-ominoes. Two n-ominoes are considered equivalent if one can be obtained from the other by translations and rotations. What is the number of distinct 15-ominoes? Your score will be equal to $25 - 13|\ln(A) - \ln(C)|$.