13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Team Round A

1. You are trying to sink a submarine. Every second, you launch a missile at a point of your choosing on the x-axis. If the submarine is at that point at that time, you sink it. A *firing sequence* is a sequence of real numbers that specify where you will fire at each second. For example, the firing sequence $2, 3, 5, 6, \ldots$ means that you will fire at 2 after one second, 3 after two seconds, 5 after three seconds, 6 after four seconds, and so on.

   (a) [**5**] Suppose that the submarine starts at the origin and travels along the positive x-axis with an (unknown) positive integer velocity. Show that there is a firing sequence that is guaranteed to hit the submarine eventually.

   (b) [**10**] Suppose now that the submarine starts at an unknown integer point on the non-negative x-axis and again travels with an unknown positive integer velocity. Show that there is still a firing sequence that is guaranteed to hit the submarine eventually.

2. [**15**] Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number or 1, at which point that player loses. For which values of N does player 1 have a winning strategy?

3. [**15**] Call a positive integer in base 10 *k-good* if we can split it into two integers y and z, such that y is all digits on the left and z is all digits on the right, and such that $y = k \cdot z$. For example, 2010 is 2-good because we can split it into 20 and 10 and $20 = 2 \cdot 10$. 20010 is also 2-good, because we can split it into 20 and 010. In addition, it is 20-good, because we can split it into 200 and 10.

   Show that there exists a 48-good perfect square.

4. [**20**] Let
$$e^x + e^y = A$$
$$xe^x + ye^y = B$$
$$x^2e^x + y^2e^y = C$$
$$x^3e^x + y^3e^y = D$$
$$x^4e^x + y^4e^y = E.$$
   Prove that if A, B, C, and D are all rational, then so is E.

5. [**20**] Show that, for every positive integer n, there exists a monic polynomial of degree n with integer coefficients such that the coefficients are decreasing and the roots of the polynomial are all integers.

6. [**20**] Let S be a convex set in the plane with a finite area a. Prove that either $a = 0$ or S is bounded. Note: a set is bounded if it is contained in a circle of finite radius. Note: a set is *convex* if, whenever two points A and B are in the set, the line segment between them is also in the set.

7. [**25**] Point P lies inside a convex pentagon $AFQDC$ such that $FPDQ$ is a parallelogram. Given that $\angle FAQ = \angle PAC = 10°$, and $\angle PFA = \angle PDC = 15°$. What is $\angle AQC$?

8. [**30**] A knight moves on a two-dimensional grid. From any square, it can move 2 units in one axis-parallel direction, then move 1 unit in an orthogonal direction, the way a regular knight moves in a game of chess. The knight starts at the origin. As it moves, it keeps track of a number t, which is initially 0. When the knight lands at the point (a, b), the number is changed from x to $ax + b$.

   Show that, for any integers a and b, it is possible for the knight to land at the points $(1, a)$ and $(-1, a)$ with t equal to b.

9. [**30**] Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be a polynomial with complex coefficients such that $a_i \neq 0$ for all i. Prove that $|r| \leq 2 \max_{i=0}^{n-1} |\frac{a_{i-1}}{a_i}|$ for all roots r of all such polynomials p. Here we let $|z|$ denote the absolute value of the complex number z.

10. Call an $2n$-digit base-10 number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$-digit number is special. (We allow leading zeros in $2n$-digit numbers).

    (a) [**20**] The sequence p_n converges to a constant c. Find c.

    (b) [**45**] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d. Find r and d.