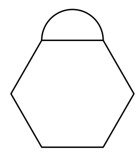
13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Team Round B

1. [**10**] How many ways are there to place pawns on an 8×8 chessboard, so that there is at most 1 pawn in each horizontal row? Express your answer in the form $p_1^{e_1} \cdot p_2^{e_2} \cdots$, where the p_i are distinct primes and the e_i are positive integers.

2. [**10**] In the following figure, a regular hexagon of side length 1 is attached to a semicircle of diameter 1. What is the longest distance between any two points in the figure?



3. [**15**] Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where each a_i is either 1 or -1. Let r be a root of p. If $|r| > \frac{15}{8}$, what is the minimum possible value of n?

4. [**20**] Find all 4-digit integers of the form $aabb$ (when written in base 10) that are perfect squares.

5. [**25**] Compute
$$\sum_{n=1}^{98} \frac{2}{\sqrt{n} + \sqrt{n+2}} + \frac{1}{\sqrt{n+1} + \sqrt{n+2}}.$$

6. [**25**] Into how many regions can a circle be cut by 10 parabolas?

7. [**30**] Evaluate
$$\sum_{k=1}^{2010} \cos^2(k)$$

8. [**30**] Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number, at which point that player loses. For how many values of N between 2 and 100 inclusive does player 1 have a winning strategy?

9. [**35**] Let S be the set of ordered pairs of integers (x, y) with $1 \leq x \leq 5$ and $1 \leq y \leq 3$. How many subsets R of S have the property that all the points of R lie on the graph of a single cubic? A cubic is a polynomial of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers (meaning that a is allowed to be 0).

10. Call an $2n$-digit number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$-digit number is special (we will allow leading zeros in the number).

    (a) [**25**] The sequence p_n converges to a constant c. Find c.

    (b) [**30**] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d. Find r and d.