13^{th} Annual Harvard-MIT Mathematics Tournament

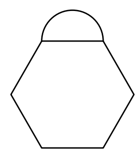
Saturday 20 February 2010

Team Round B

1. [**10**] How many ways are there to place pawns on an 8×8 chessboard, so that there is at most 1 pawn in each horizontal row? Express your answer in the form $p_1^{e_1} \cdot p_2^{e_2} \cdots$, where the p_i are distinct primes and the e_i are positive integers.

   Answer: $\boxed{3^{16}}$ If there is at most 1 pawn in each row, then each row of the chessboard may have either 0 or 1 pawn somewhere in the row. There is 1 case if there are no pawns in the row. There are 8 possible cases if there is 1 pawn in the row, one case for each square in the row. Hence for each row, there are 9 possible pawn arrangements. There are 8 rows, thus we have $9^8 = 3^{16}$.

2. [**10**] In the following figure, a regular hexagon of side length 1 is attached to a semicircle of diameter 1. What is the longest distance between any two points in the figure?



   Answer: $\boxed{\frac{1+\sqrt{13}}{2}}$ Inspection shows that one point must be on the semicircle and the other must be on the side of the hexagon directly opposite the edge with the semicircle, the bottom edge of the hexagon in the above diagram. Let O be the center of the semicircle and let M be the midpoint of the bottom edge.

   We will determine the longest distance between points in the figure by comparing the lengths of all the segments with one endpoint on the bottom edge and the other endpoint on the semicircle. Fix a point A on the bottom edge of the hexagon. Suppose that B is chosen on the semicircle such that AB is as long as possible. Let C be the circle centered at A with radius AB. If C is not tangent to the semicircle, then part of the semicircle is outside C, so we could pick a B' on the semicircle such that AB' is longer than AB. So C must be tangent to the semicircle, and AB must pass through O.

   Then OB is always $\frac{1}{2}$, no matter which A we choose on the bottom edge. All that remains is maximizing AO. This length is the hypotenuse of a right triangle with the fixed height MO, so it is maximized when AM is as large as possible - when A is an endpoint of the bottom edge. Note that $MO = 2 \cdot \frac{\sqrt{3}}{2}$, and that AM can be at most $\frac{1}{2}$, so AO can be at most $\frac{\sqrt{13}}{2}$. So the maximum distance between two points in the diagram is $AO + OB = \frac{1+\sqrt{13}}{2}$.

3. [**15**] Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where each a_i is either 1 or -1. Let r be a root of p. If $|r| > \frac{15}{8}$, what is the minimum possible value of n?

   Answer: $\boxed{4}$ We claim that $n = 4$ is the answer. First, we show that $n > 3$. Suppose that $n \leq 3$. Let r be the root of the polynomial with $|r| \geq \frac{15}{8}$. Then, by the Triangle Inequality, we have:

$$|a_n r^n| = \left|a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \ldots + a_0\right| \leq \left|a_{n-1}r^{n-1}\right| + \left|a_{n-2}r^{n-2}\right| + \ldots + |a_0|$$

$$|r|^n \leq \left|r^{n-1}\right| + \left|r^{n-2}\right| + \ldots + |1| = \frac{|r|^n - 1}{|r| - 1}$$

$$|r|^{n+1} - 2|r|^n + 1 \leq 0 \Rightarrow 1 \leq |r|^n\,(2 - |r|)$$

   The right-hand side is increasing in n, for $|r| > 1$, so it is bounded by $|r|^3(2 - |r|)$. This expression is decreasing in r for $r \geq \frac{3}{2}$. When $|r| = \frac{15}{8}$, then the right-hand side is less than 1, which violates the inequalities. Therefore $n > 3$. Now, we claim that there is a 4th degree polynomial with a root r with $|r| \geq \frac{15}{8}$. Let $p(x) = x^4 - x^3 - x^2 - x - 1$. Then $p(\frac{15}{8}) < 0$ and $p(2) > 2$. By the Intermediate Value Theorem, $p(x)$ has such a root r.

4. [**20**] Find all 4-digit integers of the form $aabb$ (when written in base 10) that are perfect squares.

Answer: $\boxed{7744}$ Let x be an integer such that x^2 is of the desired form. Then $1100a + 11b = x^2$. Then x^2 is divisible by 11, which means x is divisible by 11. Then for some integer, y, $x = 11y$. Then $1100a+11b = 11^2y^2 \Rightarrow 100a+b = 11y^2$. This means that $100a+b \equiv 0 \pmod{11} \Rightarrow a+b \equiv 0 \pmod{11}$. Because a and b must be nonzero digits, we have $2 \le a, b \le 9$, so we can write $b = 11 - a$.

Replacing b in the equation derived above, we obtain $99a + 11 = 11y^2 \Rightarrow 9a + 1 = y^2$. We check the possible values of a from 2 to 9, and only $a = 7$ yields a perfect square. When $a = 7$, $b = 4$, so the only perfect square of for $aabb$ is 7744.

5. [**25**] Compute

$$\sum_{n=1}^{98} \frac{2}{\sqrt{n}+\sqrt{n+2}} + \frac{1}{\sqrt{n+1}+\sqrt{n+2}}.$$

Answer: $\boxed{3\sqrt{11}-2\sqrt{2}+19}$ Rationalizing the denominator of both terms in the summation yields $\sqrt{n+2}-\sqrt{n}+\sqrt{n+2}-\sqrt{n+1} = 2\sqrt{n+2}-(\sqrt{n}+\sqrt{n+1})$. Then the sum $\sum_{n=1}^{98} 2\sqrt{n+2}-(\sqrt{n}+\sqrt{n+1})$ telescopes. All terms cancel except for $-(\sqrt{1}+\sqrt{2})-\sqrt{2}+2\sqrt{99}+2\sqrt{100}-\sqrt{99} = 3\sqrt{11}-2\sqrt{2}+19$.

6. [**25**] Into how many regions can a circle be cut by 10 parabolas?

Answer: $\boxed{201}$ We will consider the general case of n parabolas, for which the answer is $2n^2+1$.

We will start with some rough intuition, then fill in the details afterwards. The intuition is that, if we make the parabolas steep enough, we can basically treat them as two parallel lines. Furthermore, the number of regions is given in terms of the number of intersections of the parabolas that occur within the circle, since every time two parabolas cross a new region is created. Since two pairs of parallel lines intersect in 4 points, and pairs of parabolas also intersect in 4 points, as long as we can always make all 4 points of intersection lie inside the circle, the parallel lines case is the best we can do.

In other words, the answer is the same as the answer if we were trying to add ten pairs of parallel lines. We can compute the answer for pairs of parallel lines as follows — when we add the kth set of parallel lines, there are already $2k-2$ lines that the two new lines can intersect, meaning that each of the lines adds $2k-1$ new regions[1]. This means that we add $4k-2$ regions when adding the kth set of lines, making the answer $1+2+6+10+14+\cdots+(4n-2) = 1+2(1+3+5+7+\cdots+(2n-1)) = 1+2\cdot n^2 = 2n^2+1$.

Now that we have sketched out the solution, we will fill in the details more rigorously. First, if there are n parabolas inside the circle, and the they intersect in K points total, then we claim that the number of regions the circle is divided into will be at most $K+n+r+1$, where r is the number of parabolas that intersect the circle itself in exactly four points.

We will prove this by induction. In the base case of $n = 0$, we are just saying that the circle itself consists of exactly one region.

To prove the inductive step, suppose that we have n parabolas with K points of intersection. We want to show that if we add an additional parabola, and this parabola intersects the other parabolas in p points, then this new parabola adds either $p+1$ or $p+2$ regions to the circle, and that we get $p+2$ regions if and only if the parabola intersects the circle in exactly four points.

We will do this by considering how many regions the parabola cuts through, following its path from when it initially enters the circle to when it exits the circle for the last time. When it initially enters the circle, it cuts through one region, thereby increasing the number of regions by one[2]. Then, for each other parabola that this parabola crosses, we cut through one additional region. It is also possible for the parabola to leave and then re-enter the circle, which happens if and only if the parabola intersects

[1] This is the maximum possible number of new regions, but it's not too hard to see that this is always attainable.

[2] While the fact that a curve going through a region splits it into two new regions is intuitively obvious, it is actually very difficult to prove. The proof relies on some deep results from algebraic topology and is known as the Jordan Curve Theorem. If you are interested in learning more about this, see http://en.wikipedia.org/wiki/Jordan_curve_theorem

the circle in four points, and also adds one additional region. Therefore, the number of regions is either $p+1$ or $p+2$, and it is $p+2$ if and only if the parabola intersects the circle in four points. This completes the induction and proves the claim.

So, we are left with trying to maximize $K+n+r+1$. Since a pair of parabolas intersects in at most 4 points, and there are $\binom{n}{2}$ pairs of parabolas, we have $K \leq 4\binom{n}{2} = 2n^2 - 2n$. Also, $r \leq n$, so $K+n+r+1 \leq 2n^2+1$. On the other hand, as explained in the paragraphs giving the intuition, we can attain $2n^2+1$ by making the parabolas sufficiently steep that they act like pairs of parallel lines. Therefore, the answer is $2n^2+1$, as claimed.

7. [**30**] Evaluate

$$\sum_{k=1}^{2010} \cos^2(k)$$

Answer: $\boxed{1005 + \frac{\sin(4021)-\sin(1)}{4\sin(1)}}$ We use the identity $\cos^2(k) = \frac{1+\cos(2x)}{2}$. Then our expression evalutes to $1005 + \frac{(\cos(2)+\ldots+\cos(4020))}{2}$.

To evaluate $\cos(2)+\cdots+\cos(4020)$, let $y = \cos(2)+\cdots+\cos(4020) \Rightarrow y(\sin(1)) = \cos(2)\sin(1) + \ldots + \cos(4020)\sin(1)$. Observe that for any x, $\cos(x)\sin(1) = \frac{\sin(x+1)-\sin(x-1)}{2}$. Then $y(\sin(1)) = \frac{\sin(3)-\sin(1)}{2} + \frac{\sin(5)-\sin(3)}{2} + \ldots + \frac{\sin 4021 - \sin(4019)}{2}$. This is clearly a telescoping sum; we get $y(\sin(1)) = \frac{\sin(4021)-\sin(1)}{2}$. Then we have the desired $y = \frac{\sin(4021)-\sin(1)}{2\sin(1)}$. Then our original expression evaluates to $1005 + \frac{\sin(4021)-\sin(1)}{4\sin(1)}$.

8. [**30**] Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number, at which point that player loses. For how many values of N between 2 and 100 inclusive does player 1 have a winning strategy?

Answer: $\boxed{47}$ We claim that player 1 has a winning strategy if and only if N is even and not an odd power of 2.

First we show that if you are stuck with an odd number, then you are guaranteed to lose. Suppose you have an odd number ab, where a and b are odd numbers, and you choose to subtract a. You pass your opponent the number $a(b-1)$. This cannot be a power of 2 (otherwise a is a power of 2 and hence $a=1$, which is not allowed), so your opponent can find an odd proper divisor of $a(b-1)$ (such as a), and you will have a smaller odd number. Eventually you will get to an odd prime and lose.

Now consider even numbers that aren't powers of 2. As with before, you can find an odd proper divisor of N and pass your opponent an odd number, so you are guaranteed to win.

Finally consider powers of 2. If you have the number $N = 2^k$, it would be unwise to choose a proper divisor other than 2^{k-1}; otherwise you would give your opponent an even number that isn't a power of 2. Therefore if k is odd, you will end up with 2 and lose. If k is even, though, your opponent will end up with 2 and you will win.

Therefore player 1 has a winning strategy for all even numbers except for odd powers of 2.

9. [**35**] Let S be the set of ordered pairs of integers (x, y) with $1 \leq x \leq 5$ and $1 \leq y \leq 3$. How many subsets R of S have the property that all the points of R lie on the graph of a single cubic? A cubic is a polynomial of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers (meaning that a is allowed to be 0).

Answer: $\boxed{796}$ We observe that R must contain at most 1 point from each column of S, because no function can contain more than 1 point with the same x-coordinate. Therefore, $|R| \leq 5$ ($|R|$ denotes the number of elements of R). Note that 4 points determine a cubic, so if R is any subset of points in distinct columns and $|R| \leq 4$, then R has the desired property. There are 4^5 ways to choose at most 1 point from each column and 3^5 ways to choose exactly 1 point from each column. There are therefore $4^5 - 3^5 = 781$ subsets R of S such that $|R| \leq 4$ and all points of R lie in distinct columns. As noted, these sets all automatically have the desired property.

Now we consider all sets R of size 5. As before, each point in R must come from a different column. Let us shift our origin to $(3,2)$, and let p be the polynomial containing all 5 points of R. Then $R = \{(-2, p(-2)), (-1, p(-1)), (0, p(0)), (1, p(1)), (2, p(2))\}$.

By the method of finite differences[3], or alternately by Lagrange Interpolation[4], there is a unique polynomial p of degree less than 5 going through 5 specified points, and this polynomial is of degree less than 4 if and only if $p(-2) - 4p(-1) + 6p(0) - 4p(1) + p(2) = 0$.

Then $p(-2) + p(2) + 6p(0) = 4(p(-1) + p(1))$, where $p(-2) + p(2) \in \{-2-1, 0, 1, 2\}$, $p(-1) + p(1) \in \{-2, -1, 0, 1, 2\}$, and $p(0) \in \{-1, 0, 1\}$. We know that $6p(0)$ and $4(p(-1) + p(1))$ are necessarily even, thus we must have $p(-2) + p(2) \in \{-2, 0, 2\}$ in order for the equation to be satisfied.

Let $(a, b, c) = (p(-2) + p(2), 6p(0), 4(p(-1) + p(1)))$. The possible values of (a, b, c) that are solutions to $a + b = c$ are then $\{(-2, -6, -8), (-2, 6, 4), (0, 0, 0), (2, -6, -4), (2, 6, 8)\}$.

If $(a, b, c) = (-2, -6, -8)$, then we need $p(-2) + p(2) = -2$, $p(0) = -1$, $p(-1) + p(1) = -2$. There is only 1 possible solution to each of these equations: $(p(-2), p(2)) = (-1, -1)$ for the first one, $p(0) = -1$ for the second, and $(p(1)) = (-1, -1)$ for the third. Hence there is 1 possible subset R for the case $(a, b, c) = (-2, -6, -8)$.

If $(a, b, c) = (-2, 6, 4)$, then there is again 1 possible solution to $p(-2) + p(2) = 1$. There are two solutions to $p(-1) + p(1) = 1$: $(p(-1), p(1)) = (0, 1), (1, 0)$. Also, $p(0)$ can only be 1, so there are 2 possible subsets for this case.

If $(a, b, c) = (0, 0, 0)$, then there there are 3 possible solutions to $p(-2) + p(2) = 0$: $(p(-2), p(2)) = (-1, 1), (0, 0), (1, -1)$. Similarly, there are 3 possible solutions to $p(-1) + p(1) = 0$. Also, $p(0)$ can only be 0, so there are 9 possible subsets for this case.

If $(a, b, c) = (2, -6, -4)$, then there is 1 possible solution to $p(-2) + p(2) = 2$: $(p(-2), p(2)) = (1, 1)$. There are 2 possible solutions to $p(-1) + p(1) = -1$: $(p(-1), p(1)) = (0, -1), (-1, 0)$. Also, $p(0)$ can only be -1, so there are 2 possible subsets for this case.

If $(a, b, c) = (2, 6, 8)$, then there is 1 possible solution to $p(-2) + p(2) = 2$, as shown above. There is 1 solution to $p(-1) + p(1) = 2$: $(p(-1), p(1)) = (1, 1)$. Also, $p(0)$ can only be 1, so there is 1 possible subset for this case.

Then there are $1 + 2 + 9 + 2 + 1 = 15$ total possible subsets of size 5 that can be fit to a polynomial of degree less than 4. Hence there are $781 + 15 = 796$ possible subsets total.

10. Call an $2n$-digit number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$-digit number is special (we will allow leading zeros in the number).

    (a) **[25]** The sequence p_n converges to a constant c. Find c.

    Answer: $\boxed{\frac{1}{2}}$ We first claim that if a $2n$-digit number x has at least eight 0's and at least eight 1's and the sum of its digits is even, then x is special.

    Let A be a set of eight 0's and eight 1's and let B be the set of all the other digits. We split b arbitrily into two sets Y and Z of equal size. If $|\sum_{y \in Y} y - \sum_{z \in Z} z| > 8$, then we swap the biggest element of the set with the bigger sum with the smallest element of the other set. This transposition always decreases the absolute value of the sum: in the worst case, a 9 from the bigger set is swapped for a 0 from the smaller set, which changes the difference by at most 18. Therefore, after a finite number of steps, we will have $|\sum_{y \in Y} y - \sum_{z \in Z} z| \leq 8$.

    Note that this absolute value is even, since the sum of all the digits is even. Without loss of generality, suppose that $\sum_{y \in Y} y - \sum_{z \in Z} z$ is $2k$, where $0 \leq k \leq 4$. If we add k 0's and $8 - k$ 1's to Y, and we add the other elements of A to Z, then the two sets will balance, so x is special.

[3] See http://www.artofproblemsolving.com/Forum/weblog_entry.php?p=1263378.

[4] See http://en.wikipedia.org/wiki/Lagrange_polynomial.

(b) [**30**] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d. Find r and d.

Answer: $\boxed{(\frac{1}{4}, -1)}$ To get the next asymptotic term after the constant term of $\frac{1}{2}$, we need to consider what happens when the digit sum is even; we want to find the probability that such a number isn't balanced. We claim that the configuration that contributes the vast majority of unbalanced numbers is when all numbers are even and the sum is 2 mod 4, or such a configuration with all numbers increased by 1. Clearly this gives q_n being asymptotic to $-\frac{1}{2} \cdot 2 \cdot \left(\frac{1}{2}\right)^{2n} = -\left(\frac{1}{4}\right)^n$, so $r = \frac{1}{4}$ and $d = -1$.

To prove the claim, first note that the asymptotic probability that there are at most 4 digits that occur more than 10 times is asymptotically much smaller than $\left(\frac{1}{2}\right)^n$, so we can assume that there exist 5 digits that each occur at least 10 times. If any of those digits are consecutive, then the digit sum being even implies that the number is balanced (by an argument similar to part (a)).

So, we can assume that none of the numbers are consecutive. We would like to say that this implies that the numbers are either $0, 2, 4, 6, 8$ or $1, 3, 5, 7, 9$. However, we can't quite say this yet, as we need to rule out possibilities like $0, 2, 4, 7, 9$. In this case, though, we can just pair 0 and 7 up with 2 and 4; by using the same argument as in part (a), except using 0 and 7 both (to get a sum of 7) and 2 and 4 both (to get a sum of 6) to balance out the two sets at the end.

In general, if there is ever a gap of size 3, consider the number right after it and the 3 numbers before it (so we have $k-4, k-2, k, k+3$ for some k), and pair them up such that one pair has a sum that's exactly one more than the other (i.e. pair $k-4$ with $k+3$ and $k-2$ with k). Since we again have pairs of numbers whose sums differ by 1, we can use the technique from part (a) of balancing out the sets at the end.

So, we can assume there is no gap of size 3, which together with the condition that no two numbers are adjacent implies that the 5 digits are either $0, 2, 4, 6, 8$ or $1, 3, 5, 7, 9$. For the remainder of the solution, we will deal with the $0, 2, 4, 6, 8$ case, since it is symmetric with the other case under the transformation $x \mapsto 9 - x$.

If we can distribute the odd digits into two sets S_1 and S_2 such that (i) the differense in sums of S_1 and S_2 is small; and (ii) the difference in sums of S_1 and S_2, plus the sum of the even digits, is divisible by 4, then the same argument as in part (a) implies that the number is good.

In fact, if there are *any* odd digits, then we can use them at the beginning to fix the parity mod 4 (by adding them all in such that the sums of the two sets remain close, and then potentially switching one with an even digit). Therefore, if there are any odd digits then the number is good.

Also, even if there are no odd digits, if the sum of the digits is divisible by 4 then the number is good.

So, we have shown that almost all non-good numbers come from having all numbers being even with a digit sum that is 2 mod 4, or the analogous case under the mapping $x \mapsto 9 - x$. This formalizes the claim we made in the first paragraph, so $r = \frac{1}{4}$ and $d = -1$, as claimed.