14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

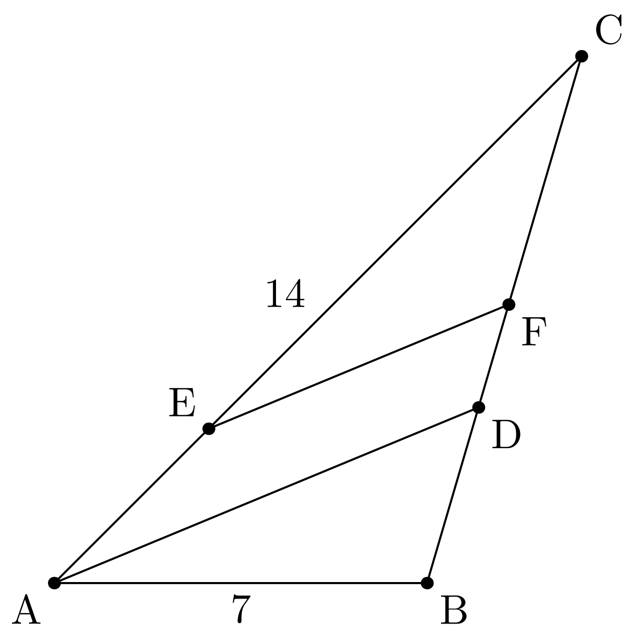
Algebra & Geometry Individual Test

1. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

   Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc, b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

   Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

([] denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$d + 7 > 14$$
$$7 + 14 > d$$

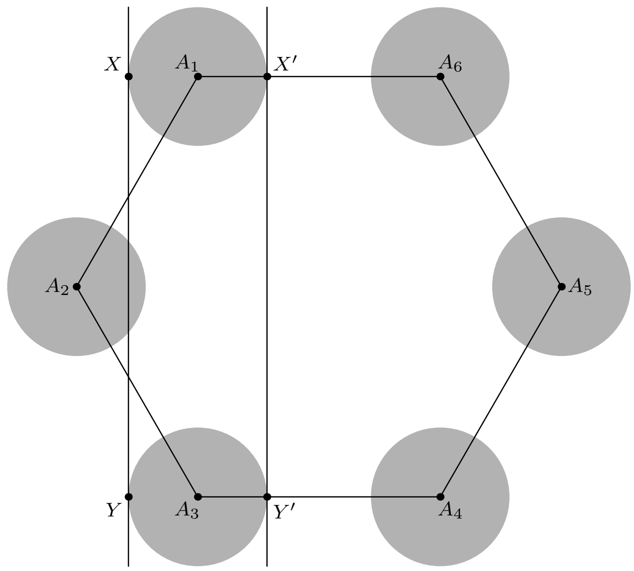
Hence $7 < d < 21$ and there are 13 possible integral values for BC.

3. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

   Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k+1$. So there are 2^{n-1} ways to get to the center square of row n.

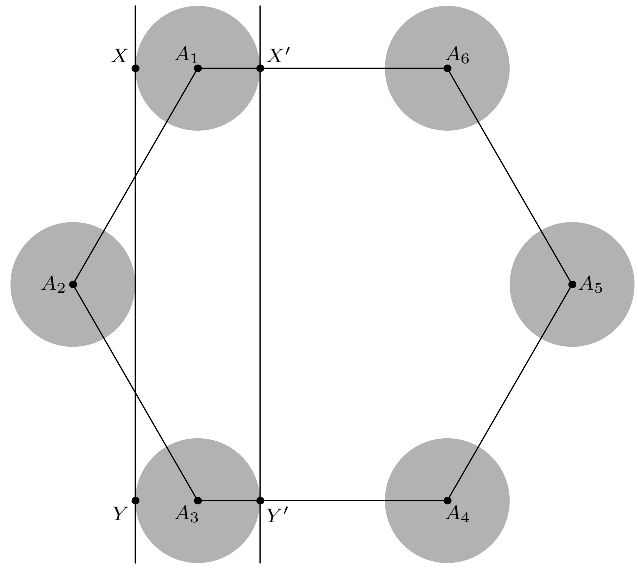
4. Let H be a regular hexagon of side length x. Call a hexagon in the same plane a "distortion" of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

   Answer: $\boxed{4}$



Let $H = A_1A_2A_3A_4A_5A_6$ be the hexagon, and for all $1 \le i \le 6$, let points A_i' be considered such that $A_iA_i' < 1$. Let $H' = A_1'A_2'A_3'A_4'A_5'A_6'$, and consider all indices modulo 6. For any point P in the plane, let $D(P)$ denote the unit disk $\{Q|PQ < 1\}$ centered at P; it follows that $A_i' \in D(A_i)$.

Let X and X' be points on line A_1A_6, and let Y and Y' be points on line A_3A_4 such that $A_1X = A_1X' = A_3Y = A_3Y' = 1$ and X and X' lie on opposite sides of A_1 and Y and Y' lie on opposite sides of A_3. If X' and Y' lie on segments A_1A_6 and A_3A_4, respectively, then segment $A_1'A_3'$ lies between the lines XY and $X'Y'$. Note that $\frac{x}{2}$ is the distance from A_2 to A_1A_3.



If $\frac{x}{2} \geq 2$, then $C(A_2)$ cannot intersect line XY, since the distance from XY to A_1A_3 is 1 and the distance from XY to A_2 is at least 1. Therefore, $A_1'A_3'$ separates A_2' from the other 3 vertices of the hexagon. By analogous reasoning applied to the other vertices, we may conclude that H' is convex.

If $\frac{x}{2} < 2$, then $C(A_2)$ intersects XY, so by choosing $A_1' = X$ and $A_3' = Y$, we see that we may choose A_2' on the opposite side of XY, in which case H' will be concave. Hence the answer is 4, as desired.

5. Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

Answer: $\boxed{101! - 1}$

We will first show that $\star$ is both commutative and associative.

- Commutativity: $a \star b = ab + a + b = b \star a$
- Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

So we need only calculate $((\ldots(1 \star 2) \star 3) \star 4) \ldots \star 100)$. We will prove by induction that

$$((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1.$$

- Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
- Inductive step:
  Suppose that
  $$(((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1.$$
  Then,

$$\begin{aligned}((((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) \star (n+1)) &= ((n+1)! - 1) \star (n+1) \\ &= (n+1)!(n+1) - (n+1) + (n+1)! - 1 + (n+1) \\ &= (n+2)! - 1\end{aligned}$$

Hence, $((\ldots(1 \star 2) \star 3) \star 4) \ldots \star n) = (n+1)! - 1$ for all n. For $n = 100$, this results to $101! - 1$.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

   Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A, will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod 7$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

   $$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6,\, l \neq k} (1 - x_l) + \frac{1}{6}.$$

   Letting $s = \sum_{l=1}^{6} x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \cdots = x_6$, and $6x_k = s$ for every k. Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

   Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

   Answer: $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x - 3)(x + 2)$. Suppose a positive integer a divides both $2x - 3$ and $x + 2$. Then a must also divide $2(x + 2) - (2x - 3) = 7$. Hence, a can either be 1 or 7. As a result, $2x - 3 = 7^n$ or -7^n for some positive integer n, or either $x + 2$ or $2x - 3$ is ± 1. We consider the following cases:

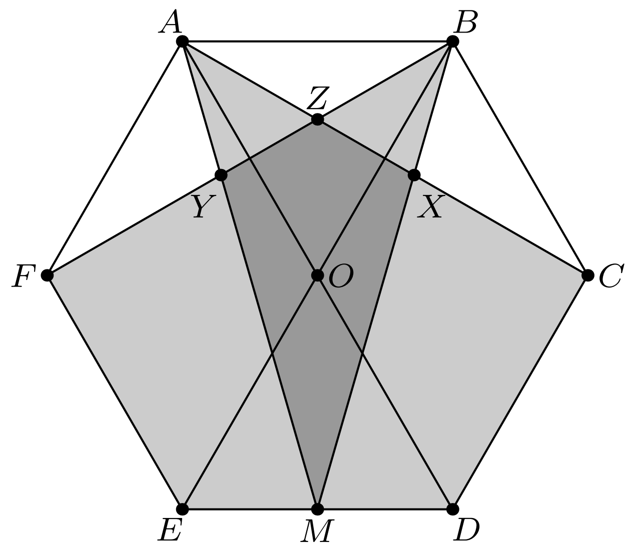
   - $(2x - 3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
   - $(2x - 3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
   - $(x + 2) = 1)$. Then $x = -1$, which yields $f(x) = -5$ not a prime power.
   - $(x + 2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
   - $(2x - 3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
   - $(2x - 3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
   - $(2x - 3) = \pm 7^n$, for $n \geq 2$. Then, since $x + 2 = \dfrac{(2x - 3) + 7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

   So x can be either -3, 2 or 5.

   (Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer $-3, 2, 5$ was marked as correct.

8. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

   Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

9. For all real numbers x, let

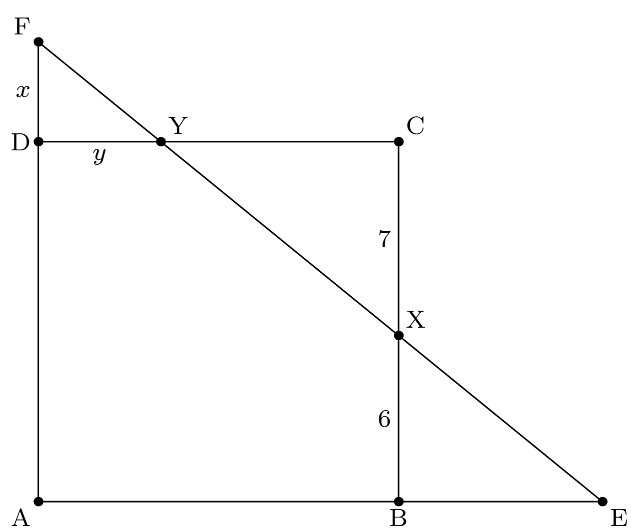
$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$

Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

Answer: $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x)) = \dfrac{\sqrt[2011]{1 - x^{2011}}}{-x}$ and $f(f(f(x))) = x$. Hence $(f(f(\ldots(f(x))\ldots))) = x$, where f is applied 2010 times. So $(f(f(\ldots(f(2011))\ldots)))^{2011} = 2011^{2011}$.

10. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD, respectively, so that the area of square $ABCD$ equals the area of triangle AEF. If EF intersects BC at X and $BX = 6$, determine DF.

Answer: $\boxed{\sqrt{13}}$



First Solution

Let Y be the point of intersection of lines EF and CD. Note that $[ABCD] = [AEF]$ implies that $[BEX] + [DYF] = [CYX]$. Since $\triangle BEX \sim \triangle CYX \sim \triangle DYF$, there exists some constant r such that $[BEX] = r \cdot BX^2$, $[YDF] = r \cdot CX^2$, and $[CYX] = r \cdot DF^2$. Hence $BX^2 + DF^2 = CX^2$, so $DF = \sqrt{CX^2 - BX^2} = \sqrt{49 - 36} = \sqrt{13}$.

Second Solution

Let $x = DF$ and $y = YD$. Since $\triangle BXE \sim \triangle CXY \sim \triangle DFY$, we have

$$\frac{BE}{BX} = \frac{CY}{CX} = \frac{DY}{DF} = \frac{y}{x}.$$

Using $BX = 6$, $XC = 7$ and $CY = 13 - y$ we get $BE = \frac{6y}{x}$ and $\frac{13-y}{7} = \frac{y}{x}$. Solving this last equation for y gives $y = \frac{13x}{x+7}$. Now $[ABCD] = [AEF]$ gives

$$\begin{aligned} 169 &= \frac{1}{2} AE \cdot AF = \frac{1}{2}\left(13 + \frac{6y}{x}\right)(13 + x). \\ 169 &= 6y + 13x + \frac{78y}{x} \\ 13 &= \frac{6x}{x+7} + x + \frac{78}{x+7} \\ 0 &= x^2 - 13. \end{aligned}$$

Thus $x = \sqrt{13}$.

11. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

    Answer: $\boxed{\frac{11-\sqrt{13}}{2}}$ Suppose f has only one distinct root r_1. Then, if x_1 is a root of $f(f(x))$, it must be the case that $f(x_1) = r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence f has two distinct roots. Let them be $r_1 \neq r_2$.

    Since $f(f(x))$ has just three distinct roots, either $f(x) = r_1$ or $f(x) = r_2$ has one distinct root. Assume without loss of generality that r_1 has one distinct root. Then $f(x) = x^2 + 6x + c = r_1$ has one root, so that $x^2 + 6x + c - r_1$ is a square polynomial. Therefore, $c - r_1 = 9$, so that $r_1 = c - 9$. So $c - 9$ is a root of f. So $(c-9)^2 + 6(c-9) + c = 0$, yielding $c^2 - 11c + 27 = 0$, or $(c - \frac{11}{2})^2 = \frac{13}{2}$. This results to $c = \frac{11 \pm \sqrt{13}}{2}$.

    If $c = \frac{11-\sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11-\sqrt{13}}{2} = (x + \frac{7+\sqrt{13}}{2})(x + \frac{5-\sqrt{13}}{2})$. We know $f(x) = \frac{-7-\sqrt{13}}{2}$ has a double root, -3. Now $\frac{-5+\sqrt{13}}{2} > \frac{-7-\sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

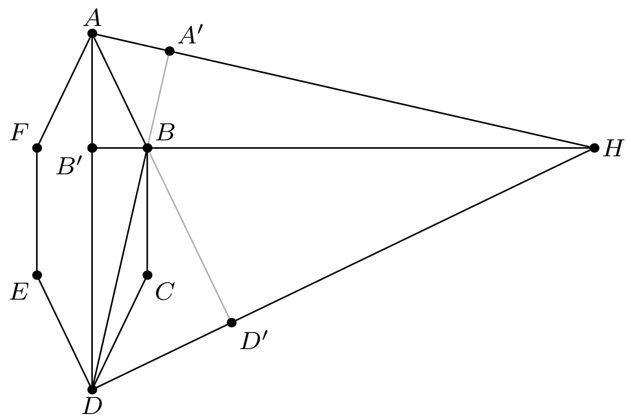
    If $c = \frac{11+\sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11+\sqrt{13}}{2} = (x + \frac{7-\sqrt{13}}{2})(x + \frac{5+\sqrt{13}}{2})$. We know $f(x) = \frac{-7+\sqrt{13}}{2}$ has a double root, -3, and this is the value of f at the vertex of the parabola, so it is its minimum value. Since $\frac{-5-\sqrt{13}}{2} < \frac{-7+\sqrt{13}}{2}$, $f(x) = \frac{-5-\sqrt{13}}{2}$ has no solutions. So in this case, f has only one real root.

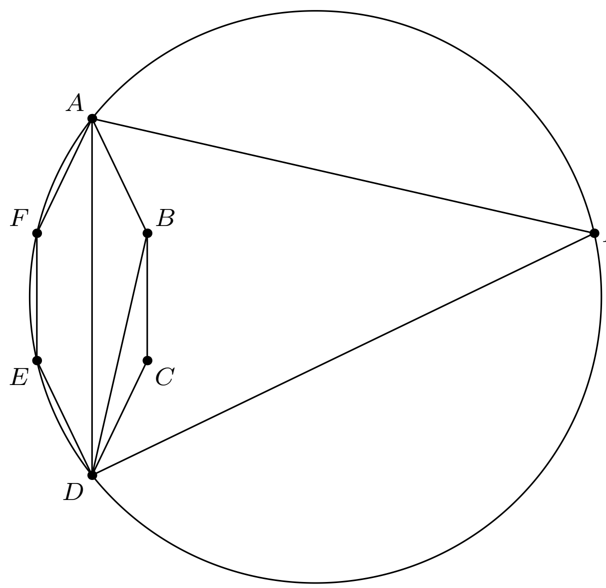
    So the answer is $c = \frac{11-\sqrt{13}}{2}$.

    Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

12. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC, AD, and EF are parallel. Let H be the orthocenter of triangle ABD. If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

    Answer: $\boxed{3}$





Note that $ABCD$ and $DEFA$ are isosceles trapezoids, so $\angle BAD = \angle CDA$ and $\angle FAD = \angle EDA$. In order for the hexagon to be convex, the angles at B, C, E, and F have to be obtuse, so $\angle A = \angle D = 4°$. Letting s be a side length of the hexagon, $AD = AB \cos\angle BAD + BC + CD\cos\angle CDA = s(1 + 2\cos\angle BAD)$, so $\angle BAD$ is uniquely determined by AD. Since the same equation holds for trapezoid $DEFA$, it follows that $\angle BAD = \angle FAD = \angle CDA = \angle EDA = 2°$. Then $\angle BCD = 180° - 2° = 178°$. Since $\triangle BCD$ is isosceles, $\angle CDB = 1°$ and $\angle BDA = 1°$. (One may also note that $\angle BDA = 1°$ by observing that equal lengths AB and BC must intercept equal arcs on the circumcircle of isosceles trapezoid $ABCD$).

Let A', B', and D' be the feet of the perpendiculars from A, B, and D to BD, DA, and AB, respectively. Angle chasing yields

$$\begin{aligned}
\angle AHD &= \angle AHB' + \angle DHB' = (90° - \angle A'AB') + (90° - \angle D'DB') \\
&= \angle BDA + \angle BAD = 1° + 2° = 3° \\
\angle HAD &= 90° - \angle AHB' = 89° \\
\angle HDA &= 90° - \angle DHB' = 88°
\end{aligned}$$

Hence the smallest angle in $\triangle HAD$ is $3°$.

It is faster, however, to draw the circumcircle of $DEFA$, and to note that since H is the orthocenter of triangle ABD, B is the orthocenter of triangle HAD. Then since F is the reflection of B across AD, quadrilateral $HAFD$ is cyclic, so $\angle AHD = \angle ADF + \angle DAF = 1° + 2° = 3°$, as desired.

13. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \le P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

Answer: $\boxed{86400000}$

For each nonnegative integer i, let $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$. (Define $x^{\underline{0}} = 1$.)

Lemma: Each polynomial with integer coefficients f can be uniquely written in the form

$$f(x) = a_n x^{\underline{n}} + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}, a_n \neq 0.$$

Proof: Induct on the degree. The base case (degree 0) is clear. If f has degree m with leading coefficient c, then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^{\underline{n}}$ can be uniquely written as $a_{n-1}x^{\underline{n-1}}(x) + \ldots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}$.

There are 120 possible choices for a_0, namely any integer in $[0, 120)$. Once $a_0, \ldots, a_{i-1}$ have been chosen so $0 \le P(0), \ldots, P(i-1) < 120$, for some $0 \le i \le 5$, then we have

$$P(i) = a_i i! + a_{i-1} i^{\underline{i-1}} + \cdots + a_0$$

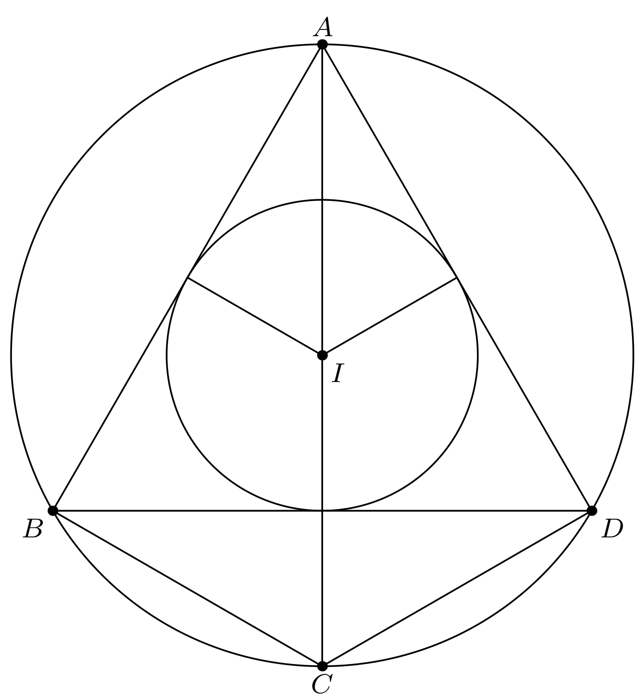
so by choosing a_i we can make $P(i)$ any number congruent to $a_{i-1}i^{\underline{i-1}} + \cdots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for a_i. Note the choice of a_i does not affect the value of $P(0), \ldots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

$$\prod_{i=0}^{5} \frac{120}{i!} = 86400000.$$

Note: Their is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the ith finite difference at the point 0.

14. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

Answer: $\boxed{2\sqrt{3}}$



Let T be the point where the incircle intersects AD, and let r be the inradius and R be the circumradius of $\triangle ABD$. Since $BC = CD = 2$, C is on the midpoint of arc BD on the opposite side of BD as A, and hence on the angle bisector of A. Thus A, I, and C are collinear. We have the following formulas:

$$\begin{aligned}
AI &= \frac{IM}{\sin \angle IAM} = \frac{r}{\sin \frac{A}{2}} \\
BC &= 2R \sin \frac{A}{2} \\
BD &= 2R \sin A
\end{aligned}$$

The last two equations follow from the extended law of sines on $\triangle ABC$ and $\triangle ABD$, respectively.

Using $AI = 2 = BC$ gives $\sin^2 \frac{A}{2} = \frac{r}{2R}$. However, it is well-known that $R \geq 2r$ with equality for an equilateral triangle (one way to see this is the identity $1 + \frac{r}{R} = \cos A + \cos B + \cos D$). Hence $\sin^2 \frac{A}{2} \leq \frac{1}{4}$ and $\frac{A}{2} \leq 30°$. Then

$$BD = 2R\left(2 \sin \frac{A}{2} \cos \frac{A}{2}\right) = BC \cdot 2 \cos \frac{A}{2} \geq 2\left(2 \cdot \frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

with equality when $\triangle ABD$ is equilateral.

Remark: Similar but perhaps simpler computations can be made by noting that if AC intersects BD at X, then $AB/BX = AD/DX = 2$, which follows from the exterior angle bisector theorem; if I_A is the A-excenter of triangle ABC, then $AI_A/XI_A = 2$ since it is well-known that C is the circumcenter of cyclic quadrilateral $BIDI_A$.

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

Answer: $\boxed{2}$

Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x, namely, for $x = g_{2i+1}, 0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x-axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c. The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned}
f\left(\frac{b+a-1}{2}\right) &< c \\
\left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a - 1 \\
\frac{1-(b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\
\frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\
4 &< (b-a-1)^2.
\end{aligned}$$

Then either $b-a-1<-2$ or $b-a-1>2$, but $b>a$, so the latter must hold and $(b-a)^2>9$. Now, the discriminant of $f(x)-x$ equals $(b-a)^2$ (the square of the difference of the two roots) and $(r_2+1)^2-4r_3$ (from the coefficients), so $(r_2+1)^2>9+4r_3$. But $r_3=g_1>g_0=0$ so $|r_2|>2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i\geq 0$ as follows. Let $N\geq 2012$ be an integer. For $\varepsilon>0$ let $h(x)=x^2-2-\varepsilon, g_\varepsilon(x)=-\sqrt{x+2+\varepsilon}$ and $G_{2N+1}=2+\varepsilon$, and define G_i recursively by $G_i=g_\varepsilon(G_{i+1}), G_{i+1}=h(G_i)$. (These two equations are consistent.) Note the following.

(i) $G_{2i}<G_{2i+1}$ and $G_{2i+1}>G_{2i+2}$ for $0\leq i\leq N-1$. First note $G_{2N}=-\sqrt{4+2\varepsilon}>-\sqrt{4+2\varepsilon+\varepsilon^2}=-2-\varepsilon$. Let l be the negative solution to $h(x)=x$. Note that $-2-\varepsilon<G_{2N}<l<0$ since $h(G_{2N})>0>G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x\geq -2-\varepsilon$, and it sends $(-2-\varepsilon,l)$ into $(l,0)$ and $(l,0)$ into $(-2-\varepsilon,l)$. It follows that the G_i, $0\leq i\leq 2N$ are well-defined; moreover, $G_{2i}<l$ and $G_{2i+1}>l$ for $0\leq i\leq N-1$ by backwards induction on i, so the desired inequalities follow.

(ii) G_i is increasing for $i\geq 2N+1$. Indeed, if $x\geq 2+\varepsilon$, then $x^2-x=x(x-1)>2+\varepsilon$ so $h(x)>x$. Hence $2+\varepsilon=G_{2N+1}<G_{2N+2}<\cdots$.

(iii) G_i is unbounded. This follows since $h(x)-x=x(x-2)-2-\varepsilon$ is increasing for $x>2+\varepsilon$, so G_i increases faster and faster for $i\geq 2N+1$.

Now define $f(x)=h(x+G_0)-G_0=x^2+2G_0x+G_0^2-G_0-2-\varepsilon$. Note $G_{i+1}=h(G_i)$ while $g_{i+1}=f(g_i)=h(g_i+G_0)-G_0$, so by induction $g_i=G_i-G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i.

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2=-2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2+\varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε, it suffices to take $\varepsilon=0$ and show that (letting $g=g_0$)

$$g^{(n)}(2)=\underbrace{g(\cdots g}_{n}(2)\cdots)\to -1 \text{ as } n\to\infty.$$

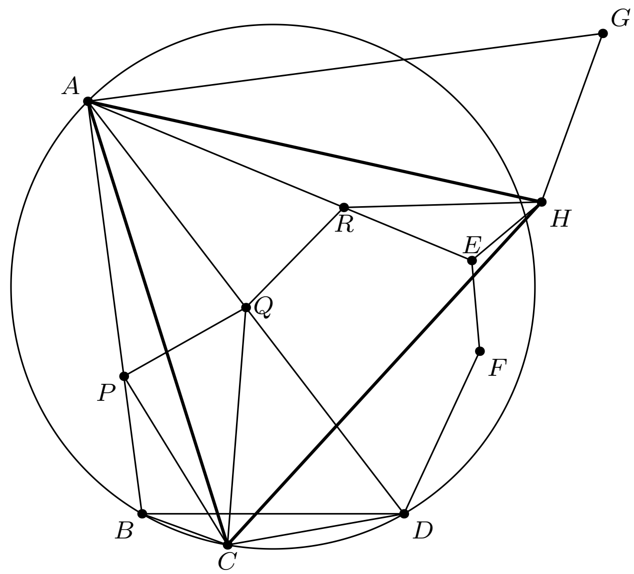
But note that for $0\leq\theta\leq\frac{\pi}{2}$,

$$g(-2\cos\theta)=-\sqrt{2-2\cos\theta}=-2\sin\left(\frac{\theta}{2}\right)=2\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right).$$

Hence by induction, $g^{(n)}(-2\cos\theta)=-2\cos\left(\frac{\pi}{2}-\frac{\pi}{4}+\cdots+(-1)^n\left(\theta-\frac{\pi}{2^n}\right)\right)$. Hence $g^{(n)}(2)=g^{(n-1)}(-2\cos 0)$ converges to $-2\cos(\frac{\pi}{2}-\frac{\pi}{4}+\cdots)=-2\cos(\frac{\pi}{3})=-1$, as needed.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP+PQ+CQ$, where P and Q may be any points lying along rays AB and AD, respectively. Determine the maximum value of m.

Answer: $\boxed{2}$



For a fixed quadrilateral $ABCD$ as described, we first show that m, the minimum possible length of $CP + PQ + QC$, equals the length of AC. Reflect B, C, and P across line AD to points E, F, and R, respectively, and then reflect D and F across AE to points G and H, respectively. These two reflections combine to give a $60°$ rotation around A, so triangle ACH is equilateral. It also follows that RH is a $60°$ rotation of PC around A, so, in particular, these segments have the same length. Because $QR = QP$ by reflection,

$$CP + PQ + QC = CQ + QR + RH.$$

The latter is the length of a broken path $CQRH$ from C to H, and by the "shortest path is a straight line" principle, this total length is at least as long as $CH = CA$. (More directly, this follows from the triangle inequality: $(CQ+QR)+RH \geq CR+RH \geq CH$). Therefore, the lower bound $m \geq AC$ indeed holds. To see that this is actually an equality, note that choosing Q as the intersection of segment CH with ray AD, and choosing P so that its reflection R is the intersection of CH with ray AE, aligns path $CQRH$ with segment CH, thus obtaining the desired minimum $m = AC$.

We may conclude that the largest possible value of m is the largest possible length of AC, namely 2: the length of a diameter of the circle.

17. Let $z = \cos \frac{2\pi}{2011} + i \sin \frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2} x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3) \ldots P(z^{2010})$.

Answer: $\boxed{2011^{2009} \cdot (1005^{2011} - 1004^{2011})}$

Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x-1) = x^{2009} + 2x^{2008} + \ldots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x-1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \ldots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$(x-1)(P(x)(x-1)+\frac{2010\cdot 2011}{2}) = x^{2010}+x^{2009}+\ldots+x-2010,$$
$$= (x^{2010}+x^{2009}+\ldots+x+1)-2011.$$

Hence,

$$P(x) = \frac{\frac{(x^{2010}+x^{2009}+\ldots+x+1)-2011}{x-1} - 2011\cdot 1005}{x-1}$$

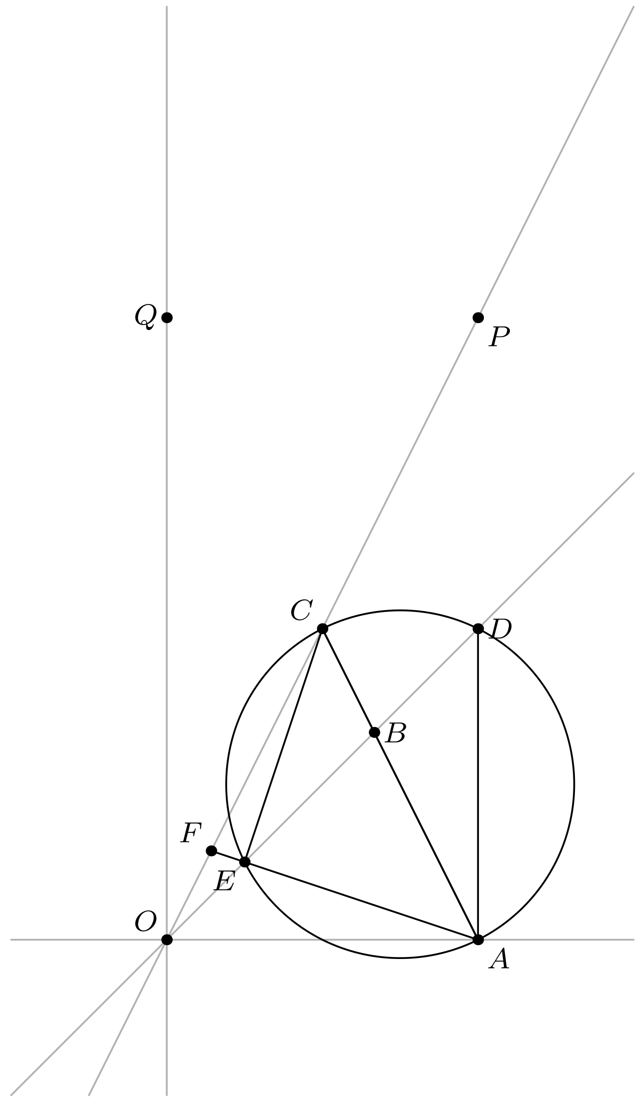
Note that $x^{2010}+x^{2009}+\ldots+x+1$ has $z, z^2, \ldots, z^{2010}$ as roots, so they vanish at those points. Plugging those 2010 powers of z into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011)\cdot 1005\cdot(x-\frac{1004}{1005})}{(x-1)^2}.$$

Note that $(x-z)(x-z^2)\ldots(x-z^{2010}) = x^{2010}+x^{2009}+\ldots+1$. Using this, the product turns out to be $2011^{2009}\cdot(1005^{2011}-1004^{2011})$.

18. Collinear points A, B, and C are given in the Cartesian plane such that $A=(a,0)$ lies along the x-axis, B lies along the line $y=x$, C lies along the line $y=2x$, and $AB/BC=2$. If $D=(a,a)$, the circumcircle of triangle ADC intersects $y=x$ again at E, and ray AE intersects $y=2x$ at F, evaluate AE/EF.

    Answer: $\boxed{7}$



Let points O, P, and Q be located at $(0,0)$, $(a,2a)$, and $(0,2a)$, respectively. Note that $BC/AB = 1/2$ implies $[OCD]/[OAD] = 1/2$, so since $[OPD] = [OAD]$, $[OCD]/[OPD] = 1/2$. It follows that $[OCD] = [OPD]$. Hence $OC = CP$. We may conclude that triangles OCQ and PCA are congruent, so $C = (a/2, a)$.
It follows that $\angle ADC$ is right, so the circumcircle of triangle ADC is the midpoint of AC, which is located at $(3a/4, a/2)$. Let $(3a/4, a/2) = H$, and let $E = (b, b)$. Then the power of the point O with respect to the circumcircle of ADC is $OD \cdot OE = 2ab$, but it may also be computed as $OH^2 - HA^2 = 13a/16 - 5a/16 = a/2$. It follows that $b = a/4$, so $E = (a/4, a/4)$.
We may conclude that line AE is $x + 3y = a$, which intersects $y = 2x$ at an x-coordinate of $a/7$. Therefore, $AE/EF = (a - a/4)/(a/4 - a/7) = (3a/4)/(3a/28) = 7$.

Remark: The problem may be solved more quickly if one notes from the beginning that lines OA, OD, OP, and OQ form a harmonic pencil because D is the midpoint of AP and lines OQ and AP are parallel.

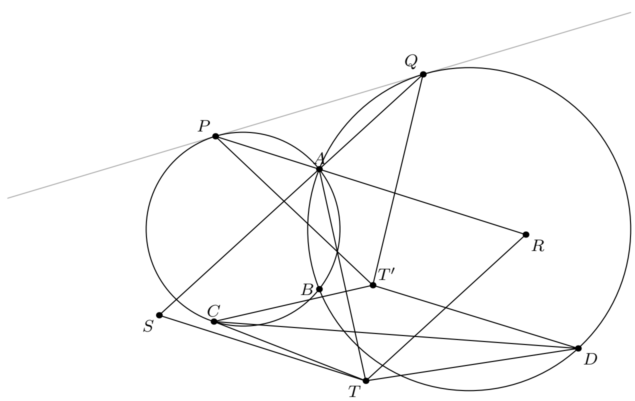
19. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n\sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4.

Answer: $\boxed{1000001100111222 \text{ and } 2211100110000012}$ Note first that $\sqrt{1 + a_n^2 + b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note that $1+a_{n+1}^2+b_{n+1}^2 = 1+a_n^2(1+a_n^2+b_n^2)+b_n^2-2a_nb_n\sqrt{1+a_n^2+b_n^2}+b_n^2(1+a_n^2+b_n^2)+a_n^2+2a_nb_n\sqrt{1+a_n^2+b_n^2} = 1+(a_n^2+b_n^2)(1+a_n^2+b_n^2)+a_n^2+b_n^2 = (1+a_n^2+b_n^2)^2$. Invoking the inductive hypothesis, we see that $\sqrt{1+a_{n+1}^2+b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.

The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_n i$ for all nonnegative integers n. It should be clear that $z_0 = 2+2i$ and $z_{n+1} = z_n(3^{2^n}+i)$. Therefore, $z_4 = (2+2i)(3^{2^0}+i)(3^{2^1}+i)(3^{2^2}+i)(3^{2^3}+i)$. This product is difficult to evaluate in the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*[1], in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2. Let $x+yi = (3^{2^0}+i)(3^{2^1}+i)(3^{2^2}+i)(3^{2^3}+i)$, and consider the balanced ternary representation of x and y. For all $0 \le j \le 15$, let x_j denote the digit in the 3^j place of x, let y_j denote the digit in the 3^j place of y, and let $b(j)$ denote the number of ones in the binary representation of j. It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod 4$, $x_j = 0$ if $b(j) \equiv 1 \pmod 2$, and $x_j = 1$ if $b(j) \equiv 0 \pmod 4$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod 4$, $y_j = 0$ if $b(j) \equiv 0 \pmod 2$, and $y_j = 1$ if $b(j) \equiv 3 \pmod 4$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B. Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B. Let points R and S lie along rays PA and QA, respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q. Let O be the circumcenter of triangle ASR, and let C and D be the midpoints of major arcs AP and AQ, respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

Answer: $\boxed{142.5}$



We use directed angles throughout the solution.

Let T denote the point such that $\angle TCD = 1/2\angle APQ$ and $\angle TDC = 1/2\angle AQP$. We claim that T is the circumcenter of triangle SAR.

Since $CP = CA$, $QP = RA$, and $\angle CPQ = \angle CPA + \angle APQ = \angle CPA + \angle ACP = \angle CAR$, we have $\triangle CPQ \cong \triangle CAR$. By spiral similarity, we have $\triangle CPA \sim \triangle CQR$.

Let T' denote the reflection of T across CD. Since $\angle TCT' = \angle APQ = \angle ACP$, we have $\triangle TCT' \sim \triangle ACP \sim \triangle RCQ$. Again, by spiral similarity centered at C, we have $\triangle CTR \sim \triangle CT'Q$. But $CT = CT'$, so $\triangle CTR \cong \triangle CT'Q$ and $TR = T'Q$. Similarly, $\triangle DTT' \sim \triangle DAQ$, and spiral similarity centered at D shows that $\triangle DTA \cong \triangle DT'Q$. Thus $TA = T'Q = TR$.

We similarly have $TA = T'P = TS$, so T is indeed the circumcenter. Therefore, we have $\angle COD = \angle CTD = 180^\circ - \frac{45^\circ}{2} - \frac{30^\circ}{2} = 142.5^\circ$.

[1] http://en.wikipedia.org/wiki/Balanced_ternary