15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Combinatorics Test

1. In the game of Minesweeper, a number on a square denotes the number of mines that share at least one vertex with that square. A square with a number may not have a mine, and the blank squares are undetermined. How many ways can the mines be placed in this configuration?

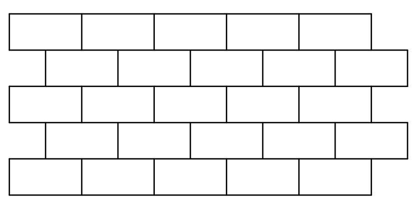
	2		1		2

Answer: $\boxed{95}$ Let A be the number of mines in the first two columns. Let B, C, D, E be the number of mines in the third, fourth, fifth, and sixth columns, respectively. We need to have $A + B = 2$, $B + C + D = 1$, and $D + E = 2$. This can happen in three ways, which are $(A, B, C, D, E) = (2, 0, 1, 0, 2), (2, 0, 0, 1, 1), (1, 1, 0, 0, 2)$. This gives $(10)(2)(1) + (10)(3)(2) + (5)(3)(1) = 95$ possible configurations.

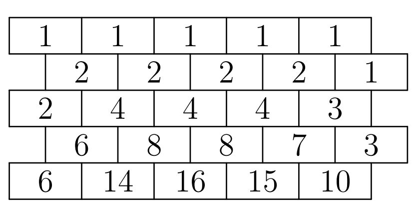
2. Brian has a 20-sided die with faces numbered from 1 to 20, and George has three 6-sided dice with faces numbered from 1 to 6. Brian and George simultaneously roll all their dice. What is the probability that the number on Brian's die is larger than the sum of the numbers on George's dice?

Answer: $\boxed{\frac{19}{40}}$ Let Brian's roll be d and let George's rolls be x, y, z. By pairing the situation d, x, y, z with $21 - d, 7 - x, 7 - y, 7 - z$, we see that the probability that Brian rolls higher is the same as the probability that George rolls higher. Given any of George's rolls x, y, z, there is exactly one number Brian can roll which will make them tie, so the probability that they tie is $\frac{1}{20}$. So the probability that Brian wins is $\frac{1-\frac{1}{20}}{2} = \frac{19}{40}$.

3. In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



Answer: $\boxed{61}$ The number of valid selections is equal to the number of paths which start at a top brick and end at a bottom brick. We compute these by writing 1 in each of the top bricks and letting lower bricks be the sum of the one or two bricks above them. Thus, the number inside each brick is the number of paths from that brick to the top. The bottom row is 6, 14, 16, 15, 10, which sums to 61.



4. A frog is at the point $(0, 0)$. Every second, he can jump one unit either up or right. He can only move to points (x, y) where x and y are not both odd. How many ways can he get to the point $(8, 14)$?

Answer: $\boxed{330}$ When the frog is at a point (x, y) where x and y are both even, then if that frog chooses to move right, his next move will also have to be a step right; similarly, if he moves up, his next move will have to be up.

If we "collapse" each double step into one step, the problem simply becomes how many ways are there to move to the point $(4, 7)$ using only right and up steps, with no other restrictions. That is 11 steps total, so the answer is $\binom{11}{4} = 330$.

5. Dizzy Daisy is standing on the point $(0, 0)$ on the xy-plane and is trying to get to the point $(6, 6)$. She starts facing rightward and takes a step 1 unit forward. On each subsequent second, she either takes a step 1 unit forward or turns 90 degrees counterclockwise then takes a step 1 unit forward. She may never go on a point outside the square defined by $|x| \le 6, |y| \le 6$, nor may she ever go on the same point twice. How many different paths may Daisy take?

   Answer: $\boxed{131922}$ Because Daisy can only turn in one direction and never goes to the same square twice, we see that she must travel in an increasing spiral about the origin. Clearly, she must arrive at $(6, 6)$ coming from below. To count her paths, it therefore suffices to consider the horizontal and vertical lines along which she travels (out of 5 choices to move upward, 6 choices leftward, 6 choices downward, and 6 choices rightward). Breaking up the cases by the number of complete rotations she performs, the total is $\binom{5}{0}\binom{6}{0}^3 + \binom{5}{1}\binom{6}{1}^3 + \binom{5}{2}\binom{6}{2}^3 + \binom{5}{3}\binom{6}{3}^3 + \binom{5}{4}\binom{6}{4}^3 + \binom{5}{5}\binom{6}{5}^3 = 131922$.

6. For a permutation σ of $1, 2, \ldots, 7$, a *transposition* is a swapping of two elements. (For instance, we could apply a transposition to the permutation $3, 7, 1, 4, 5, 6, 2$ and get $3, 7, 6, 4, 5, 1, 2$ by swapping the 1 and the 6.)

   Let $f(\sigma)$ be the minimum number of transpositions necessary to turn σ into the permutation $1, 2, 3, 4, 5, 6, 7$. Find the sum of $f(\sigma)$ over all permutations σ of $1, 2, \ldots, 7$.

   Answer: $\boxed{22212}$ To solve this problem, we use the idea of a *cycle* in a permutation. If σ is a permutation, we say that $(a_1\ a_2\ \cdots\ a_k)$ is a cycle if $\sigma(a_i) = \sigma(a_{i+1})$ for $1 \le i \le k-1$ and $\sigma(a_k) = a_1$. Any permutation can be decomposed into disjoint cycles; for instance, the permutation $3, 7, 6, 4, 5, 1, 2$, can be written as $(1\ 3\ 6)(2\ 7)(4)(5)$. For a permutation σ, let $g(\sigma)$ be the number of cycles in its cycle decomposition. (This includes single-element cycles.)

   Claim. For any permutation σ on n elements, $f(\sigma) = n - g(\sigma)$.

   Proof. Given a cycle $(a_1\ a_2\ \cdots\ a_k)$ (with $k \ge 2$) of a permutation σ, we can turn this cycle into the identity permutation with $k-1$ transpositions; first we swap a_1 and a_2; that is, we replace σ with a permutation σ' such that instead of $\sigma(a_k) = a_1$ and $\sigma(a_1) = a_2$, we have $\sigma'(a_k) = a_2$ and $\sigma'(a_1) = a_1$. Now, σ' takes a_1 to itself, so we are left with the cycle $(a_2\ \cdots\ a_n)$. We continue until the entire cycle is replaced by the identity, which takes $k-1$ transpositions. Now, for any σ, we resolve each cycle in this way, making a total of $n - g(\sigma)$ transpositions, to turn σ into the identity permutation.

   This shows that $n - g(\sigma)$ transpositions; now let us that we cannot do it in less. We show that whenever we make a transposition, the value of $n - g(\sigma)$ can never decrease by more than 1. Whenever we swap two elements, if they are in different cycles, then those two cycles merge into one; thus $n - g(\sigma)$ actually increased. If the two elements are in one cycle, then the one cycle splits into two cycles, so $n - g(\sigma)$ decreased by only one, and this proves the claim.

   Thus, we want to find

   $$\sum_{\sigma \in S_7} (7 - g(\sigma)) = 7 \cdot 7! - \sum_{\sigma \in S_7} g(\sigma)$$

   to evaluate the sum, we instead sum over every cycle the number of permutations it appears in. For any $1 \le k \le 7$, the number of cycles of size k is $\frac{n!}{(n-k)!k}$, and the number of permutations each such cycle can appear in is $(n-k)!$. Thus we get that the answer is

   $$7 \cdot 7! - \sum_{k=1}^{7} \frac{7!}{k} = 22212.$$

7. You are repeatedly flipping a fair coin. What is the expected number of flips until the first time that your previous 2012 flips are 'HTHT...HT'?

   Answer: $\boxed{(2^{2014}-4)/3}$ Let S be our string, and let $f(n)$ be the number of binary strings of length n which do not contain S. Let $g(n)$ be the number of strings of length n which contain S but whose prefix of length $n-1$ does not contain S (so it contains S for the "first" time at time n).

   Consider any string of length n which does not contain S and append S to it. Now, this new string contains S, and in fact it must contain S for the first time at either time $n+2$, $n+4$, ..., or $n+2012$. It's then easy to deduce the relation

$$f(n) = g(n+2) + g(n+4) + \cdots + g(n+2012)$$

   Now, let's translate this into a statement about probabilities. Let t be the first time our sequence of coin flips contains the string S. Dividing both sides by 2^n, our equality becomes

$$P(t > n) = 4P(t = n+2) + 16P(t = n+4) + \cdots + 2^{2012}P(t = n+2012)$$

   Summing this over all n from 0 to ∞, we get

$$\sum P(t > n) = 4 + 16 + \cdots + 2^{2012} = (2^{2014}-4)/3$$

   But it is also easy to show that since t is integer-valued, $\sum P(t > n) = E(t)$, and we are done.

8. How many ways can one color the squares of a 6x6 grid red and blue such that the number of red squares in each row and column is exactly 2?

   Answer: $\boxed{67950}$ Assume the grid is $n \times n$. Let $f(n)$ denote the number of ways to color exactly two squares in each row and column red. So $f(1) = 0$ and $f(2) = 1$. We note that coloring two squares red in each row and column partitions the set $1, 2, \ldots, n$ into cycles such that i is in the same cycle as, and adjacent to, j iff column i and column j have a red square in the same row. Each i is adjacent to two other, (or the same one twice in a 2-cycle).

   Now consider the cycle containing 1, and let it have size k. There are $\binom{n}{2}$ ways to color two squares red in the first column. Now we let the column that is red in the same row as the top ball in the first column, be the next number in the cycle. There are $n-1$ ways to pick this column, and $n-2$ ways to pick the second red square in this column (unless $k = 2$). Then there are $(n-2)(n-3)$ ways to pick the red squares in the third column. and $(n-j)(n-j+1)$ ways to pick the jth ones for $j \leq k-1$. Then when we pick the kth column, the last one in the cycle, it has to be red in the same row as the second red square in column 1, so there are just $n-k+1$ choices. Therefore if the cycle has length k there are
   $\frac{n(n-1)}{2} \times (n-1)(n-2) \times \ldots \times (n-k+1)(n-k+2) \times (n-k+1)$ ways, which equals: $\frac{n!(n-1)!}{2(n-k)!(n-k)!}$.
   Summing over the size of the cycle containing the first column, we get

$$f(n) = \sum_{k=2}^{n} \frac{1}{2} f(n-k) \frac{(n)!(n-1)!}{(n-k)!(n-k)!}$$

$$\frac{2nf(n)}{n!n!} = \sum_{k=2}^{n} \frac{f(n-k)}{(n-k)!(n-k)!}$$

$$\frac{2nf(n)}{n!n!} - \frac{2(n-1)f(n-1)}{(n-1)!(n-1)!} = \frac{f(n-2)}{(n-2)!(n-2)!}$$

   We thus obtain the recursion:

$$f(n) = n(n-1)f(n-1) + \frac{n(n-1)^2}{2} f(n-2)$$

Then we get:

$f(1) = 0$

$f(2) = 1$

$f(3) = 6$

$f(4) = 12 \times 6 + 18 = 90$

$f(5) = 20 \times 90 + 40 \times 6 = 2040$

$f(6) = 30 \times 2040 + 75 \times 90 = 67950.$

9. A parking lot consists of 2012 parking spots equally spaced in a line, numbered 1 through 2012. One by one, 2012 cars park in these spots under the following procedure: the first car picks from the 2012 spots uniformly randomly, and each following car picks uniformly randomly among all possible choices which maximize the minimal distance from an already parked car. What is the probability that the last car to park must choose spot 1?

   Answer: $\boxed{\frac{1}{2062300}}$ We see that for 1 to be the last spot, 2 must be picked first (with probability $\frac{1}{n}$), after which spot n is picked. Then, cars from 3 to $n-1$ will be picked until there are only gaps of 1 or 2 remaining. At this point, each of the remaining spots (including spot 1) is picked uniformly at random, so the probability that spot 1 is chosen last here will be the reciprocal of the number of remaining slots.

   Let $f(n)$ denote the number of empty spots that will be left if cars park in $n+2$ consecutive spots whose ends are occupied, under the same conditions, except that the process stops when a car is forced to park immediately next to a car. We want to find the value of $f(2009)$. Given the gap of n cars, after placing a car, there are gaps of $f(\lfloor \frac{n-1}{2} \rfloor)$ and $f(\lceil \frac{n-1}{2} \rceil)$ remaining. Thus, $f(n) = f(\lfloor \frac{n-1}{2} \rfloor) + f(\lceil \frac{n-1}{2} \rceil)$. With the base cases $f(1) = 1, f(2) = 2$, we can determine with induction that $f(x) = \begin{cases} x - 2^{n-1} + 1 & \text{if } 2^n \le x \le \frac{3}{2} \cdot 2^n - 2, \\ 2^n & \text{if } \frac{3}{2} \cdot 2^n - 1 \le x \le 2 \cdot 2^n - 1. \end{cases}$

   Thus, $f(2009) = 1024$, so the total probability is $\frac{1}{2012} \cdot \frac{1}{1024+1} = \frac{1}{2062300}$.

10. Jacob starts with some complex number x_0 other than 0 or 1. He repeatedly flips a fair coin. If the n^{th} flip lands heads, he lets $x_n = 1 - x_{n-1}$, and if it lands tails he lets $x_n = \frac{1}{x_{n-1}}$. Over all possible choices of x_0, what are all possible values of the probability that $x_{2012} = x_0$?

    Answer: $\boxed{1, \frac{2^{2011}+1}{3 \cdot 2^{2011}}}$ Let $f(x) = 1 - x$, $g(x) = \frac{1}{x}$. Then for any x, $f(f(x)) = x$ and $g(g(x)) = x$. Furthermore, $f(g(x)) = 1 - \frac{1}{x}$, $g(f(g(x))) = \frac{x}{x-1}$, $f(g(f(g(x)))) = \frac{1}{1-x}$, $g(f(g(f(g(x))))) = 1 - x = f(x)$, so for all n, x_n is one of $x, \frac{1}{x}, 1 - \frac{1}{x}, \frac{x}{x-1}, \frac{1}{1-x}, 1 - x$, and we can understand the coin flipping procedure as moving either left or right with equal probability along this cycle of values.

    For most x, all six of these values are distinct. In this case, suppose that we move right R times and left $2012 - R$ times between x_0 and x_{2012}. For $x_{2012} = x_0$, we need to have that $R - 2012 + R \equiv 0 \pmod 6$, or $R \equiv 1 \pmod 3$. The number of possible ways to return to x_0 is then $a = \binom{2012}{1} + \binom{2012}{4} + \cdots + \binom{2012}{2011}$. Let $b = \binom{2012}{0} + \binom{2012}{3} + \cdots + \binom{2012}{2010} = \binom{2012}{2} + \binom{2012}{5} + \cdots + \binom{2012}{2012}$. Then we have $a + 2b = 2^{2012}$ and that $b = \frac{(1+1)^{2012} + (1+\omega)^{2012} + (1+\omega^2)^{2012}}{3}$, where ω is a primitive third root of unity. It can be seen that $1 + \omega$ is a primitive sixth root of unity and $1 + \omega^2$ is its inverse, so $(1+\omega)^{2012} = (1+\omega)^2 = \omega$, and similarly $(1+\omega^2)^{2012} = \omega^2$. Therefore, $b = \frac{\cdot 2^{2012} - 1}{3}$, so $a = 2^{2012} - 2b = \frac{2^{2012}+2}{3}$, and our desired probability is then $\frac{a}{2^{2012}} = \frac{2^{2012}+2}{3 \cdot 2^{2012}} = \frac{2^{2011}+1}{3 \cdot 2^{2011}}$.

    For some x_0, however, the cycle of values can become degenerate. It could be the case that two adjacent values are equal. Let y be a value that is equal to an adjacent value. Then $y = \frac{1}{y}$ or $y = 1 - y$, which gives $y \in \{-1, \frac{1}{2}\}$. Therefore, this only occurs in the cycle of values $-1, 2, \frac{1}{2}, \frac{1}{2}, 2, -1$. In this case, note that after 2012 steps we will always end up an even number of steps away from our starting point, and each of the numbers occupies two spaces of opposite parity, so we would need to return to our original location, just as if all six numbers were distinct. Therefore in this case we again have that the probability that $x_{2012} = x_0$ is $\frac{2^{2011}+1}{3 \cdot 2^{2011}}$.

It is also possible that two numbers two apart on the cycle are equal. For this to be the case, let y be the value such that $f(g(y)) = y$. Then $1 - \frac{1}{x} = x$, or $x - 1 = x^2$, so $x = \frac{1 \pm i\sqrt{3}}{2}$. Let $\zeta = \frac{1+i\sqrt{3}}{2}$. Then we get that the cycle of values is $\zeta, \overline{\zeta}, \zeta, \overline{\zeta}, \zeta, \overline{\zeta}$, and since at the end we are always an even number of spaces away from our starting location, the probability that $x_{2012} = x_0$ is 1.

Finally, we need to consider the possibility that two opposite numbers are equal. In this case we have a y such that $f(g(f(y))) = y$, or $\frac{x}{x-1} = x$, so $x = 2$. In this case we obtain the same cycle of numbers in the case where two adjacent numbers are equal, and so we again obtain the probability $\frac{2^{2011}+1}{3 \cdot 2^{2011}}$. Therefore, the only possibilities are $1, \frac{2^{2011}+1}{3 \cdot 2^{2011}}$.